Positronium Decay : Gauge Invariance and Analyticity

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Abstract

The construction of positronium decay amplitudes is handled through the use of dispersion relations. In this way, emphasis is put on basic QED principles: gauge invariance and soft-photon limits (analyticity).

A firm grounding is given to the factorization approaches, and some ambiguities in the spin and energy structures of the positronium wavefunction are removed. Non-factorizable amplitudes are naturally introduced. Their dynamics is described, especially regarding the enforcement of gauge invariance and analyticity through delicate interferences. The important question of the completeness of the present theoretical predictions for the decay rates is then addressed. Indeed, some of those non-factorizable contributions are unaccounted for by NRQED analyses. However, it is shown that such new contributions are highly suppressed, being of $\mathcal{O}\left(\alpha^3\right)$.

Finally, a particular effective form factor formalism is constructed for para-positronium, allowing a thorough analysis of binding energy effects and analyticity implementation.

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1 Introduction

Positronium is a bound state made of an electron with a positron. In this paper, we will be interested in the ground states, parapositronium (p-Ps, singlet, $J = 0$) and orthopositronium (o-Ps, triplet, $J = 1$). Positronium is unstable, it decays when its components annihilate into photons. The lifetimes are quite different for the two species, because parity and charge conjugation conservations imply that p-Ps decays into an even number of photons, while o-Ps can only decay into an odd number.

Positronium was discovered in 1951 [1]. Since then, many measurements of the lifetime of both species have been achieved. The most precise to date are

$$\Gamma^{exp}\left(p\text{-}Ps \to 2\gamma\right) = 7.9909\,(17) \times 10^9\,\mathrm{sec}^{-1} \;\; [2] \tag{1}$$

$$\Gamma^{exp}\left(o\text{-}Ps \to 3\gamma\right) = \begin{cases} 7.0398\,(29) \;\; \mu sec^{-1} & \text{Tokyo (SiO}_2\text{ Powder)[3]} \\ 7.0514\,(14) \;\; \mu sec^{-1} & \text{Ann Arbor (Gas)[4]} \\ 7.0482\,(16) \;\; \mu sec^{-1} & \text{Ann Arbor (Vacuum)[5]} \end{cases} \tag{2}$$

Obviously, the situation is not clear for the o-Ps lifetime, since results are mutually exclusive. This is known as the orthopositronium lifetime puzzle.

The corresponding lowest order theoretical predictions were first obtained as

$$\Gamma_{p\text{-}Ps} = \frac{\alpha^5 m}{2} = 8.033 \times 10^9\,\mathrm{sec}^{-1} \qquad [6] \tag{3a}$$

$$\Gamma_{o\text{-}Ps} = \alpha^6 m \frac{2\left(\pi^2 - 9\right)}{9\pi} = 7.211 \;\; \mu\,\mathrm{sec}^{-1} \;\; [7] \tag{3b}$$

with α the fine structure constant and m the electron mass. A great deal of work has been done on the calculation of radiative corrections, which include perturbative QED corrections to a non-relativistic treatment of the bound state wavefunction. The present highly accurate theoretical predictions for the respective lifetimes are (see for example [8] to [15]):

$$\Gamma_{p\text{-}Ps} = \frac{\alpha^5 m}{2}\left(1 + \delta_{2\gamma} + \delta_{4\gamma}\right) = 7.989620\,(13) \times 10^9\,\mathrm{sec}^{-1} \tag{4a}$$

$$\Gamma_{o\text{-}Ps} = \alpha^6 m \frac{2\left(\pi^2 - 9\right)}{9\pi}\left(1 + \delta_{3\gamma} + \delta_{5\gamma}\right) = 7.039965\,(10) \;\; \mu\,\mathrm{sec}^{-1} \tag{4b}$$

The corrections to the dominant 2γ and 3γ modes are given by

$$\begin{aligned} \delta_{2\gamma} &= -A_p \frac{\alpha}{\pi} + 2\alpha^2 \ln\frac{1}{\alpha} + B_p \left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi}\ln^2\frac{1}{\alpha} + C_p \frac{\alpha^3}{\pi}\ln\frac{1}{\alpha} \\ &\approx -0.0058828 + 0.0005242 + 0.0000277 - 0.0000045 - 0.0000048 \\ \delta_{3\gamma} &= -A_o \frac{\alpha}{\pi} - \frac{\alpha^2}{3}\ln\frac{1}{\alpha} + B_o \left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi}\ln^2\frac{1}{\alpha} + C_o \frac{\alpha^3}{\pi}\ln\frac{1}{\alpha} \\ &\approx -0.0238939 - 0.0000873 + 0.0002402 - 0.0000045 + 0.0000034 \end{aligned}$$

with coefficients

$$\begin{array}{ll} A_p = 5 - \pi^2/4 & A_o = 10.286606\,(10) \\ B_p = 5.14\,(30) & B_o = 44.52\,(26) \\ C_p = -7.919\,(1) & C_o = 5.517\,(1) \end{array}$$

The non-logarithmic $\mathcal{O}\left(\alpha^2\right)$ corrections $B_{p,o}$ and logarithmic $\mathcal{O}\left(\alpha^3 \ln\alpha\right)$ corrections $C_{p,o}$ have been obtained only recently [12]. The small contributions to the lifetimes from multi-photon decay modes are [16]

$$\begin{aligned} \delta_{4\gamma} &= 0.274\,(1)\left(\frac{\alpha}{\pi}\right)^2 \\ \delta_{5\gamma} &= 0.19\,(1)\left(\frac{\alpha}{\pi}\right)^2 \end{aligned}$$

The four-photon mode increases the p-Ps width by about $1 \times 10^5\,\mathrm{sec}^{-1}$, while the five-photon mode contributes for $7 \times 10^{-6}\ \mu\mathrm{sec}^{-1}$ to the o-Ps width.

As can be observed, agreement between theory and experiment is good for parapositronium, but the B_p, C_p radiative corrections are not yet accessible experimentally. For orthopositronium, these corrections render the theoretical prediction still closer to the experimental measurement of Ref. [3] (anyway, beyond discriminating between experimental results, B_o and C_o radiative corrections are not tested either).

Positronium is a test ground for bound state treatments in Quantum Field Theory. The first try dates back to the 40's, with decay rates expressed through a factorized formula [6]

$$\Gamma\left(p\text{-}Ps \rightarrow 2\gamma\right) = \left|\phi_o\right|^2 \cdot \left(4v_{rel}\sigma\left(e^+e^- \rightarrow 2\gamma\right)\right)_{v_{rel}\rightarrow 0} \tag{5}$$

with ϕ_o the Schrödinger positronium fundamental state wavefunction at the origin, $\sigma\left(e^+e^- \rightarrow 2\gamma\right)$ the total cross section for $e^+e^- \rightarrow 2\gamma$ and v_{rel} the relative velocity of e^+ and e^- in their center-of-mass frame. Since then, more sophisticated decay amplitudes have been constructed, and systematic procedures for calculating corrections have been developed (like non-relativistic QED (NRQED), see for example [17]), but factorization remains central. To avoid misunderstanding, let us stress that by factorization is usually meant a positronium decay amplitude built as a convolution integral,

$$\mathcal{M}\left(p\text{-}Ps \rightarrow 2\gamma\right) \sim \int d^3\mathbf{k}\ \psi\left(\mathbf{k}\right) \cdot \mathcal{M}\left(e^+\left(-\mathbf{k}\right)e^-\left(\mathbf{k}\right) \rightarrow 2\gamma\right) \tag{6}$$

with $\psi\left(\mathbf{k}\right)$ the momentum Schrödinger wavefunction and $\mathcal{M}\left(e^+e^- \rightarrow 2\gamma\right)$ the scattering amplitude for on-shell electron-positron into photons.

Several questions concerning this formalism need to be addressed:

– *The basic factorization* of the bound state dynamics from the annihilation process remains as the basic postulate. For low order corrections, this approximation is unquestionable, but for $\mathcal{O}\left(\alpha^2\right)$ corrections, factorization has to be tested. Indeed, non-perturbative phenomena responsible for the off-shellness of the electron and positron inside the positronium are of $\mathcal{O}\left(\alpha^2\right)$ (since positronium mass minus twice the electron mass is of that order). In other words, to get a sensible theoretical prediction at $\mathcal{O}\left(\alpha^2\right)$, one must carefully analyze how binding energy effects enter the general factorization approach.

– *Violation of energy conservation ?* Positronium being a bound state, $M < 2m$ (M the positronium mass and m the electron mass). The total energy of the on-shell electron-positron pair entering the scattering process is $2\sqrt{\mathbf{k}^2 + m^2}$, clearly greater than M, and getting worse as $\mathbf{k}$ increases in the integration.

– The enforcement of *gauge invariance* may be problematic. This is more technical. Let us say that, in general, one has to project the electron-positron pair into a spin state compatible with the total spin of the bound state. This bound state spin state is usually treated non-relativistically, introducing some potential threads to the gauge invariance of the decay amplitude.

– Some further problems concern the *Bethe-Salpeter wavefunction.* There are a lot of technicalities and subtleties associated with the bound state wavefunction. This wavefunction can in principle be obtained covariantly from the Bethe-Salpeter (BS) equation. Usually, one has to introduce some approximations to solve this equation, ending with a non-covariant four-dimensional wavefunction. This $4D$ wavefunction is then reduced to $3D$ by making additional approximations for its temporal part (energy-dependent part). This last reduction may be questionable. (and, as said above, non-covariant wavefunctions can be difficult to accommodate with gauge invariance).

– Last but not least, *analytical behavior in the soft photon limit* may be ill-defined. By this we mean that since factorization treats intermediate charged states as on-shell, infrared singularities are introduced. This problem is well-known, but some confusion seems to creep in the literature. We stress that: **For a model to be correct, it is not sufficient to show that IR divergences are successfully disposed of**. From general quantum field theory principles, a decay amplitude involving only neutral external bosons has a correct analytical behaviour if and only if this amplitude *vanishes* in the soft photon limit (not simply be finite). This is a very profound principle, at the root of electrodynamics [18].

The purpose of the present paper is to answer those questions using dispersion relations (DR). As will become clear in the course of the presentation, DR appear as the natural framework to study positronium decay amplitudes. Some questions on the applicability of DR analyses to bound state decays could be raised, but, in our view, the fact that all the questions above find consistent answers gives great confidence in this approach.

The most prominent achievements of DR analyses are

1. a well-defined systematic procedure for factorizing the bound state dynamics from the annihilation process,

2. an identification of the dynamics ensuring both gauge invariance and the correct soft photon limits.

The basic idea of our approach is to consider the positronium decay amplitudes through a fully relativistic loop model, where only off-shell constituents appear. The nice surprise is that standard three-dimensional convolution-type amplitudes like (6) are recovered through a DR. Contrary to the usual approaches, no approximations are needed to reach the three-dimensional form. Most importantly, we will show that many variants of the standard formula involve some unnecessary and dangerous (i.e. gauge-dependent) approximations in their treatment of spins, and we will remove them. Since our derivation relies on well-established techniques of quantum field theory, we conclude that some $\mathcal{O}\left(\alpha^2\right)$ corrections have been forgotten in those works.

It is when analytical properties are analyzed that our procedure gains its full respectability. We will see that DR introduce some additional non-factorizable (i.e. non-convolution-type) contributions to the decay amplitudes. Further, some bound state structure-dependent non-factorizable contributions must be introduced. All those additional contributions interfere with the standard amplitude contributions to enforce both gauge invariance and the correct analytical properties. Interestingly, because the present approach relies on a dynamical characterization of the various contributions, it permits an evaluation of their magnitudes. This is an important fact because if those additional contributions prove to have been missed in current calculations, the positronium lifetime $\mathcal{O}\left(\alpha^2\right)$ corrections would be wrong. We will not be able to provide a definite answer to this question, but nevertheless point towards an optimistic view. Provided no pathological enhancement occurs, the NRQED result indeed should contain all $\mathcal{O}\left(\alpha^2\right)$ corrections.

The paper is articulated in three parts. The first part contains the presentation of the general results, and discussions about their implications. After a short review of the standard approaches to the construction of positronium decay amplitudes, we go on to prove that our model reproduces those standard decay amplitudes. This result is completely general. We then apply the insight gained to the analysis of soft-photon behavior, and find the mechanism at play to ensure the correct analytical limits.

The remaining of the paper contains applications and illustrations. We analyze in some details parapositronium decay into two photons in the second part of the paper. This decay is a bit special because the kinematics forbids soft photon energies to vanish. Also, the appearance of a pseudoscalar coupling (parapositronium is a pseudoscalar) has interesting consequences, allowing form factor-like dispersion relations to be built. This technique is then used in various calculations. This section closes by the description of an improved basis for the perturbation series (4a), where binding energy effects are singled out.

The final part contains discussions about soft-photon limits. We take as an example the decay of paradimuonium (the $\mu^+\mu^-$ bound state, p-Dm) into $e^+e^-\gamma$. This is the simplest positronium-like decay process where a photon can have a vanishing energy. Again, the pseudoscalar property of paradimuonium simplifies the discussion, allowing a thorough analysis of the non-factorizable contributions. We also briefly discuss the decay $K_S \to e^+e^-\gamma$, modeled by an intermediate charged pion loop. The reason is the striking similarity between this "elementary" particle decay and the bound state analogue p-$Dm \to e^+e^-\gamma$. For example, exactly the same mechanism is responsible for the implementation of the correct soft-photon limits (SPL). It is very interesting to see Quantum Field Theory at play in transforming soft-photon singularities from intermediate charged states into a *vanishing* SPL, no matter the details of the specific models. Also, we hope that this incursion into the realm of kaon physics will convince the most sceptical readers that our approach is viable and correct, as it is based on general quantum field theory principles.

NB : This paper has emerged from three previous works, Ref. [19].

2 Positronium Decay Amplitudes and Dispersion Relations

Standard amplitudes for the decay of parapositronium ($J=0$) or orthopositronium ($J=1$) are defined as (see for example [11], [12], [20])

$$\begin{aligned}\mathcal{M}\left({}^{2J+1}S \to A\right) &= \sqrt{2M}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\psi(\mathbf{k})\frac{1}{\sqrt{2E_\mathbf{k}}}\frac{1}{\sqrt{2E_\mathbf{k}}} \\ &\quad\times\mathcal{M}\left(e^-(\mathbf{k},\xi), e^+(-\mathbf{k},\xi') \to A\right)_{(\xi,\xi')={}^{2J+1}S} \qquad (7)\end{aligned}$$

where M is the bound state mass and $\psi(\mathbf{k})$ is the Schrödinger bound state wavefunction. This wavefunction is taken in a convolution integral with the scattering amplitude for on-shell $e^+e^- \to A$, constrained to the required spin state. By working out this constraint, we reach the commonly quoted positronium decay amplitude (hereafter denoted 'standard approach' amplitude)

$$\mathcal{M}\left({}^{2J+1}S \to A\right) = \sqrt{2M}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\psi(\mathbf{k})\frac{m}{2E_\mathbf{k}}Tr\left\{\frac{1+\gamma^0}{\sqrt{2}}\mathbf{P}\Gamma\left(e^-(\mathbf{k}), e^+(-\mathbf{k}) \to A\right)\right\} \qquad (8)$$

where

$$\mathcal{M}\left(e^-(\mathbf{k},\xi), e^+(-\mathbf{k},\xi') \to A\right) = \overline{v}(-\mathbf{k},\xi')\ \Gamma\left(e^-(\mathbf{k}), e^+(-\mathbf{k}) \to A\right)\ u(\mathbf{k},\xi)$$

$\mathbf{P} = \gamma^5$ for parapositronium, $\not e$ for orthopositronium with polarization vector e^μ. Finally, the width is calculated as

$$\Gamma\left({}^{2J+1}S \to A\right) = \frac{1}{S}\frac{1}{2J+1}\frac{1}{2M}\int d\Phi_A \left|\mathcal{M}\left({}^{2J+1}S \to A\right)\right|^2 \tag{9}$$

with S a symmetry factor for the final state A.

This standard decay formula has many peculiar features, and needs a rigorous grounding. As quoted in the introduction, factorization and SPL properties are the most pressing problems. In section 2.1, we intend to concentrate on the first point, leaving analytical questions to the following one. To do so we will particularize the discussion to parapositronium decay to 2γ, which cannot suffer from any pathological SPL due to its kinematics.

Let us now introduce our model. Basically, we assume a loop structure for the decay amplitudes. In other words, positronium decays into a virtual electron-positron pair which subsequently annihilates into real or virtual photons (an odd number for ortho-states, an even number for para-states). The coupling of the positronium to its constituents is described by a form factor, denoted by F_B. It is not assumed to be a constant, since a constant form factor would amount to consider positronium as a point-like bound state.

For parapositronium decay into two photons, our model is represented in figure 1. The corresponding amplitude is

$$\mathcal{M}\left(p\text{-}Ps \to \gamma\gamma\right) = \int \frac{d^4q}{\left(2\pi\right)^4} F_B Tr\left\{\gamma_5 \frac{i}{\not q - \frac{1}{2}\not P - m}\Gamma^{\mu\nu}\frac{i}{\not q + \frac{1}{2}\not P - m}\right\}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^* \tag{10}$$

with P the positronium four-momentum and $F_B \equiv F_B\left(q^2, P\cdot q\right)$ the form factor. The tensor $\Gamma^{\mu\nu}$ is the scattering amplitude for off-shell e^+e^-, with incoming momenta $\frac{1}{2}P - q$ and $\frac{1}{2}P + q$, into two photons :

$$\begin{aligned} \Gamma^{\mu\nu} &= \Gamma^{\mu\nu}\left(e^+\left(\tfrac{1}{2}P - q\right)e^-\left(\tfrac{1}{2}P + q\right) \to \gamma\left(l_1\right)\gamma\left(l_2\right)\right) \\ &= ie\gamma^\mu \frac{i}{\not q - \frac{1}{2}\not P + \not l_1 - m} ie\gamma^\nu + ie\gamma^\nu \frac{i}{\not q + \frac{1}{2}\not P - \not l_1 - m} ie\gamma^\mu \end{aligned} \tag{11}$$

Remarks :

1) The model is extended to orthopositronium decays through the replacement of γ_5 by $\not e$.

2) The electron and positron in the loop are never on-shell, because $M < 2m$.

3) F_B contains all the information about the bound state. Let us postulate a form for this coupling as (in the positronium center-of-mass frame)

$$F_B \equiv C\phi_0\mathcal{F}\left(q_o, \mathbf{q}^2\right)\left(\mathbf{q}^2 + \gamma^2\right) \tag{12}$$

with C a constant, and $\gamma^2 = m^2 - M^2/4$ related to the binding energy $E_B = 2m - M$. In QED, E_B and γ^2 are related to the fine structure constant as $\gamma^2 \approx m^2\alpha^2/4$ and $E_B = -m\alpha^2/4$.

2.1 The loop model reproduces standard decay amplitudes

We can state our result in three steps (details are found in the appendix).

First, we define

$$\operatorname{Im}\mathcal{T}\left(P^2\right) \equiv \operatorname{Im}\mathcal{M}\left(p\text{-}Ps\left(P^2\right) \to \gamma\gamma\right) \tag{13}$$

with the absorptive part given by the two vertical cuts, figure 2.

Then, using an unsubstracted dispersion relation (see [21], [22]) with $s = P^2$:

$$\mathcal{T}\left(M^2\right) = \mathrm{Re}\,\mathcal{T}\left(M^2\right) = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds}{s-M^2}\,\mathrm{Im}\,\mathcal{T}\left(s\right) \tag{14}$$

($\mathcal{T}\left(M^2\right) = \mathrm{Re}\,\mathcal{T}\left(M^2\right)$ because $M < 2m$), and changing of variables, one recovers a factorized form (i.e. a convolution type amplitude):

$$\mathcal{M}\left(p\text{-}Ps \rightarrow \gamma\gamma\right) = \frac{C}{2}\int\frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\frac{\phi_0\mathcal{F}\left(0,\mathbf{k}^2\right)}{2E_{\mathbf{k}}}Tr\left\{\gamma_5\left(m-\not{k}'\right)\Gamma\left(m+\not{k}\right)\right\} \tag{15}$$

with Γ given by $\Gamma^{\mu\nu}\left(e^-\left(k\right), e^+\left(k'\right) \rightarrow \gamma\gamma\right)\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*$ and $k = \left(E_k, \mathbf{k}\right),\ k' = \left(E_k, -\mathbf{k}\right)$, i.e. the same Γ as in (8).

Finally, if one further identifies $C = \sqrt{M}/m$ and $\phi_0\mathcal{F}\left(0,\mathbf{k}^2\right) = \psi\left(\mathbf{k}\right)$, by neglecting the $\mathbf{k}$ dependence in the projectors $(m-\not{k}')$ and $(m+\not{k})$, (15) reduces to (8).

As a corollary, the energy-dependence of the form factor is demonstrated to be irrelevant. Indeed, the dispersion relation fixes $q_o = 0$. From now on, we will simplify the notation and identify $\mathcal{F}\left(0,\mathbf{k}^2\right) = \mathcal{F}\left(\mathbf{k}^2\right)$.

2.1.1 Discussion

Let us now comment on this result. Interpreting standard approach amplitudes as dispersion relation integrals, we can give answers to most of the questions raised in the introduction.

First, factorization appears simply as a manifestation of the optical theorem: the appearance of on-shell intermediate states is expected in the imaginary part. The dispersive integral "shifts" them off-shell: the amplitude at the physical point $s = M^2$ is purely real, i.e. only off-shell e^+e^- circulate inside the loop. Also, the apparent non-conservation of energy is explained, since the dispersive integral is done along the loop model imaginary part cut, where the initial energy is indeed sufficient to get on-shell constituents. In other words, the standard approach amplitudes are constructed like in "old-fashioned perturbation theory". In the context of quantum field theory, their natural framework is dispersion theory.

Second, the spin projections are treated covariantly. The projector often seen in the literature is an approximation. In those approaches, the projector $\left(1+\gamma_0\right)\mathbf{P}$ comes from the spin wavefunction of the positronium, while here it comes directly from the loop propagators. This is more adequate since it is the moving electron-positron entering the scattering process that must be constrained to the required spin state. The projectors appearing in (15) for particles in motion will introduce some new corrections to the positronium decay rate of the order of the binding energy γ^2, i.e. α^2 (here "new" means relatively to the result found starting from (8) or any equivalent formula).

Further, the covariant projectors have the important property of preserving gauge invariance. Indeed, when using $\left(1+\gamma_0\right)\mathbf{P}$ as projector, gauge invariance is valid only if the constituents are taken at rest, i.e. $\mathbf{k} \equiv 0$ (the so-called static limit). In other words, (15) is a gauge invariant positronium decay amplitude while (8) is not.

Another consequence is that since the loop model appears at the root of standard approaches, interesting alternative calculational tools can be built from it. In particular, it provides for an easy procedure to integrate the wavefunction exactly, keeping tracks of binding energy effects even at lowest order. This is the subject of section 3.

2.1.2 A comment about Bethe-Salpeter

Viewing the decay process through a loop model is far from new. After all, it is the approach followed using Bethe-Salpeter (BS) analyses, where one starts with

$$\mathcal{M}\left(Ps \rightarrow n\gamma\right) = \int \frac{d^4q}{\left(2\pi\right)^4} Tr\left\{\Gamma\left(e^- e^+ \rightarrow n\gamma\right)\Psi\left(q\right)\right\} \tag{16}$$

with Γ the off-shell scattering amplitude (given in (11) for $n = 2$). The loop structure emerges from the BS identification of the wavefunction as

$$\Psi\left(q\right) = \frac{i}{\frac{1}{2}\not{P} + \not{q} - m}\Gamma_{BS}\left(\frac{P}{2} + q, \frac{P}{2} - q, P\right)\frac{i}{\frac{1}{2}\not{P} - \not{q} - m}$$

with Γ_{BS} the BS vertex function. All this is standard textbook material (see for example [23]).

As is well-known, the BS equation cannot be solved exactly, and one has to rely on some approximations. The most popular one for positronium is that of Barbieri-Remiddi [24], which can be viewed as a four-dimensional generalization of the usual Schrödinger wavefunction. To make contact with the three-dimensional integral representation for the decay amplitude (7), the wavefunction is approximated as $\Psi\left(q\right) \sim \delta\left(q_o\right)\psi\left(\mathbf{q}\right)$ in (16) (see for example [11], [15]).

The novelty of our approach is to use dispersion relations to study the BS loop. This has great advantages over approximate methods, because our treatment is *exact*. Also, all the previous comments show the insights gained doing the reduction our way. In fact, starting with a four-dimensional BS form factor or vertex function $\Gamma_{BS} \sim F_B\left(q_0, \mathbf{q}\right)$, *the dispersion relations alone enforce* $q_0 = 0$, i.e. the energy-dependence is set to zero.

Another particularity of our approach is the treatment of spin, since we took the BS vertex Γ_{BS} with its spin structure replaced by γ_5 or $\not{\varepsilon}$ ($F_B\left(q_0, \mathbf{q}\right)$ is a scalar), a replacement dictated by the bound state behavior under parity and charge conjugation. This allows us to identify the correct covariant spin projectors for the e^+e^- pair, and to preserve manifest gauge invariance.

In conclusion, the present dispersion technique could help clarify some problems concerning the reduction of four-dimensional BS wavefunctions to three-dimensional ones (note that one should not confuse the present reduction of the BS *wavefunction* with the $4D \rightarrow 3D$ reduction of the BS *equation* itself, extensively discussed in the literature, see for example [25] and references quoted there).

2.2 Analytical Properties of Positronium Decay Amplitudes

We have just seen that the standard positronium decay amplitudes are given a natural interpretation in terms of dispersion relation integrals. Let us see the implications for orthopositronium decay into three photons.

The loop model amplitude consists of six diagrams (figure 3)

$$\begin{aligned}\mathcal{M}\left(o\text{-}Ps \rightarrow \gamma\gamma\gamma\right) &= e_\alpha\left(P\right)\int \frac{d^4q}{\left(2\pi\right)^4} F_B \times \varepsilon^*_{1\mu}\varepsilon^*_{2\nu}\varepsilon^*_{3\rho} \\ &\quad \times Tr\left\{\gamma^\alpha \frac{i}{\not{q} - \frac{1}{2}\not{P} - m}\Gamma^{\mu\nu\rho}\frac{i}{\not{q} + \frac{1}{2}\not{P} - m}\right\}\end{aligned} \tag{17}$$

where the off-shell scattering amplitudes are

$$\begin{aligned}\Gamma^{\mu\nu\rho} &\equiv \Gamma^{\mu\nu\rho}\left(e^+(\tfrac{1}{2}P - q)e^-(\tfrac{1}{2}P + q) \rightarrow \gamma\left(l_1\right)\gamma\left(l_2\right)\gamma\left(l_3\right)\right) \\ &= -ie^3\left(\Gamma^{\mu\nu\rho}_{12} + \Gamma^{\rho\nu\mu}_{32} + \Gamma^{\rho\mu\nu}_{31}\right)\end{aligned} \tag{18}$$

with

$$\Gamma_{ij}^{\mu\nu\rho} = \gamma^{\nu}\frac{1}{\not{q}-\frac{1}{2}\not{P}+\not{l}_j-m}\gamma^{\rho}\frac{1}{\not{q}+\frac{1}{2}\not{P}-\not{l}_i-m}\gamma^{\mu} + \gamma^{\mu}\frac{1}{\not{q}-\frac{1}{2}\not{P}+\not{l}_i-m}\gamma^{\rho}\frac{1}{\not{q}+\frac{1}{2}\not{P}-\not{l}_j-m}\gamma^{\nu}$$

and $P = l_1 + l_2 + l_3$. To compute these loop amplitudes, we start by extracting the imaginary part, and we discover a completely different picture than the one described earlier for the two-photon parapositronium decay.

Since the discussion is rather involved, the following summary of what we intend to show may be useful:

1. There are six vertical cut diagrams $\mathcal{V}$ (figure 4a)*, with the properties:*

– reproduce the standard convolution-type decay amplitude;

– are separately gauge invariant;

– are badly behaved in the soft-photon limit, i.e. violation of analyticity.

2. There are additional contributions arising from the oblique cuts $\mathcal{D}$ (figure 4b)*, with the properties:*

– are not of the factorized convolution-type;

– are not gauge invariant by themselves, except for a constant F_B;

– also badly behaved in the IR; the combination $\mathcal{V}+\mathcal{D}$ also fails to vanish in the SPL, except for a constant F_B.

3. *In addition to the loop model amplitudes* $\mathcal{V}$ and $\mathcal{D}$*, structure-dependent contributions* $\mathcal{S}$ *must be considered* (figure 4c)*, with the properties:*

– are non factorizable, requiring a new form factor for the vertex $Ps\ e^+ e^-\ \gamma$;

– act to restore gauge invariance;

– interfere with both $\mathcal{V}$ and $\mathcal{D}$ to ensure the correct soft-photon limits.

(if F_B is constant, structure terms need not be considered)

4. *The various contributions should scale as*

$$\frac{\mathcal{D}}{\mathcal{V}} \sim O\left(\alpha^2\right),\ \frac{\mathcal{S}}{\mathcal{V}} \sim O\left(\alpha^3\right)$$

We will spend some time to justify and explain these results, reviewing each point in turn. The strategy is to use dispersion relations as a classification scheme for contributions to the decay process, and fundamental principles (gauge invariance and soft-photon limits) to characterize and constraint each type of contributions.

Let us state our results more precisely, introducing some notations. We write the decay amplitude

$$\begin{aligned} \mathcal{M}\left(o\text{-}Ps\left(M^2\right)\to\gamma\gamma\gamma\right) &= \left(\mathcal{M}^{\mu\nu\rho}\left(\text{loop model}\right)+\mathcal{M}^{\mu\nu\rho}\left(\text{structure}\right)\right)\varepsilon_{1\mu}^{*}\varepsilon_{2\nu}^{*}\varepsilon_{3\rho}^{*} \\ &\equiv \left(\mathcal{V}^{\mu\nu\rho}+\mathcal{D}^{\mu\nu\rho}+\mathcal{S}^{\mu\nu\rho}\right)\varepsilon_{1\mu}^{*}\varepsilon_{2\nu}^{*}\varepsilon_{3\rho}^{*} \\ &\equiv \mathcal{V}+\mathcal{D}+\mathcal{S} \end{aligned}$$

Using the optical theorem, each contribution is expressed as

$$\mathcal{V} = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds}{s-M^2}\,\mathrm{Im}\,\mathcal{V}\left(s\right) \qquad (19a)$$

$$\mathcal{D} = \frac{1}{\pi}\sum_{i=1}^{3}\int_{s_{\min}(l_i)}^{\infty}\frac{ds}{s-M^2}\operatorname{Im}\mathcal{D}_i(s) \tag{19b}$$

$$\mathcal{S} = \frac{1}{\pi}\sum_{i=1}^{3}\int_{s_{\min}(l_i)}^{\infty}\frac{ds}{s-M^2}\operatorname{Im}\mathcal{S}_i(s) \tag{19c}$$

with the vertical cuts imaginary part

$$\operatorname{Im}\mathcal{V}\left(P^2\right) = \varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\varepsilon_{3\rho}^*\int d\Phi_2\mathcal{M}\left(o\text{-}Ps\left(P^2\right)\to e^+e^-\right)\times\mathcal{M}_{scatt}^{\mu\nu\rho}\left(e^+e^-\to\gamma_{l_1}\gamma_{l_2}\gamma_{l_3}\right) \tag{20}$$

The integration is over the two-body e^+e^- phase-space $d\Phi_2$, and summation over spins is understood. Oblique and structure-dependent imaginary parts are

$$\operatorname{Im}\mathcal{D}_1\left(P^2\right) = \varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\varepsilon_{3\rho}^*\int d\Phi_2\mathcal{M}_{IB;1}^{\mu}\left(o\text{-}Ps\left(P^2\right)\to e^+e^-\gamma_{l_1}\right)\times\mathcal{M}_{scatt;23}^{\nu\rho}\left(e^+e^-\to\gamma_{l_2}\gamma_{l_3}\right) \tag{21}$$

$$\operatorname{Im}\mathcal{S}_1\left(P^2\right) = \varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\varepsilon_{3\rho}^*\int d\Phi_2\mathcal{M}_{SD;1}^{\mu}\left(o\text{-}Ps\left(P^2\right)\to e^+e^-\gamma_{l_1}\right)\times\mathcal{M}_{scatt;23}^{\nu\rho}\left(e^+e^-\to\gamma_{l_2}\gamma_{l_3}\right) \tag{22}$$

and similarly for photon $2,3$. The oblique cut amplitudes $\mathcal{M}_{IB}$ are interpreted as bremsstrahlung amplitudes, hence the subscript.

Gauge invariance is expressed through Ward identities. Let us concentrate on the photon l_1 for definiteness. First note that scattering amplitudes are all gauge invariant

$$\begin{aligned} l_{1,\mu}\mathcal{M}_{scatt}^{\mu\nu\rho}\left(e^+e^-\to\gamma_{l_1}\gamma_{l_2}\gamma_{l_3}\right) &= 0 \\ l_{1,\mu}\mathcal{M}_{scatt;12(3)}^{\mu\nu}\left(e^+e^-\to\gamma_{l_1}\gamma_{l_{2(3)}}\right) &= 0 \end{aligned}$$

because the e^+e^- are on-shell. This implies gauge invariance for the vertical cuts

$$l_{1,\mu}\mathcal{V}^{\mu\nu\rho} = 0$$

On the other hand, IB and SD amplitudes cannot be separately gauge invariant, since $l_{1,\mu}\mathcal{M}_{IB;1}^{\mu}\neq 0$. This serves as a constraint for SD amplitudes: we impose

$$l_{1,\mu}\mathcal{M}_{IB;1}^{\mu} + l_{1,\mu}\mathcal{M}_{SD;1}^{\mu} = 0$$

Provided this last equality holds, the positronium decay amplitude is gauge invariant.

To analyze the soft-photon behaviors, let us expand each amplitude around $l_1^0 = 0$:

$$\begin{aligned} \mathcal{V} &\overset{l_1^0\sim 0}{=} \frac{v_1}{l_1^0} + v_2 + \mathcal{O}\left(l_1^0\right) \\ \mathcal{D} &\overset{l_1^0\sim 0}{=} \frac{d_1}{l_1^0} + d_2 + \mathcal{O}\left(l_1^0\right) \\ \mathcal{S} &\overset{l_1^0\sim 0}{=} s_2 + \mathcal{O}\left(l_1^0\right) \end{aligned}$$

Selection rules alone suffice to cancel IR divergences, i.e. $v_1 = d_1 = 0$, and

$$\mathcal{M}\left(o\text{-}Ps\left(M^2\right)\to\gamma_{l_1}\gamma_{l_2}\gamma_{l_3}\right) \overset{l_1^0\sim 0}{\sim} \text{Constant}$$

The implementation of the correct SPL requires in addition

$$v_2 + d_2 + s_2 = 0$$

which guarantees

$$\mathcal{M}\left(o\text{-}Ps\left(M^2\right)\to\gamma_{l_1}\gamma_{l_2}\gamma_{l_3}\right)\overset{l_1^0\sim 0}{\sim}\mathcal{O}\left(l_1^0\right)$$

Similar behaviors are expected for photon $2,3$.

Let us now prove those assertions, and discuss their interpretations.

2.2.1 Factorization properties

By an immediate application of the result of section 2.1, one can easily show that *vertical cuts alone reproduce the standard three-dimensional convolution-type amplitude*:

$$\begin{aligned}\mathcal{V} &= \frac{C}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\varphi_0\mathcal{F}\left(\mathbf{k}^2\right)}{2E_{\mathbf{k}}}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\varepsilon_{3\rho}^* \\ &\quad\times Tr\left\{\not{\epsilon}\left(m-\not{k}'\right)\Gamma^{\mu\nu\rho}\left(e^+\left(k'\right)e^-\left(k\right)\to\gamma\gamma\gamma\right)\left(m+\not{k}\right)\right\}\end{aligned}\tag{23}$$

with on-shell e^+e^-. Note that it is important to treat spin projectors covariantly to preserve gauge invariance (if $(m+\not{k})\not{\epsilon}(m-\not{k}')$ is approximated as $(1+\gamma_0)\not{\epsilon}$, gauge invariance is lost, except in the static limit).

Bremsstrahlung amplitudes generate a factorization problem for oblique cuts $\mathcal{D}$. In fact, the emission of a photon by either the electron or the positron breaks the "symmetry" of the diagram (by symmetry is meant a configuration in which each lepton is carrying half the positronium momentum). Let us recall the implications of this symmetry for the vertical cuts. The first term in (20) is

$$\mathcal{M}\left(o\text{-}Ps\left(P^2\right)\to e^+\left(\tfrac{P}{2}-q\right)e^-\left(\tfrac{P}{2}+q\right)\right)\sim F_B\left(q-\tfrac{1}{2}P,q+\tfrac{1}{2}P\right)\tag{24}$$

The energy conservation at that vertex and the on-shell condition for the e^+e^- (in the $o\text{-}Ps$ center-of-mass frame) amount to

$$\delta\left(q_0\right)\delta\left(|\mathbf{q}|-\sqrt{P^2/4-m^2}\right)$$

This allows us to write F_B very simply as $F_B=C\phi_0\mathcal{F}\left(\mathbf{q}^2\right)\left(\mathbf{q}^2+\gamma^2\right)$ (12). Such simplifications do not occur for the oblique cuts, the energy-dependence of the form factor is not set to zero but to a complicated $\mathbf{q}$-dependent value. We refer to the calculation of oblique cuts for $p\text{-}Dm\to e^+e^-\gamma$ (section 4) for an explicit illustration of the problem.

2.2.2 Gauge Invariance and Low's theorem

Low's theorem states that *an amplitude involving only neutral external bosons must vanish in the soft photon limit*, i.e. when anyone photon's energy is taken to zero.

This theorem must apply to positronium decay amplitudes. One must then conclude that standard amplitudes do not exhibit the correct analytical behavior, since individual scattering amplitudes contained in $\Gamma\left(e^-\left(\mathbf{k}\right),e^+\left(-\mathbf{k}\right)\to\gamma\gamma\gamma\right)$ explode in the SPL. One can view this simply by noting the presence in Γ of electron propagators of the form

$$\gamma_\mu\frac{i}{\not{k}-\not{l}_i-m}\gamma_\nu\ \cdots$$

and when the ith photon energy l_i^0 is vanishing, the propagator explodes because the constituents are on-shell. In fact, the situation is improved by a consequence of selection rules: when the six vertical cut amplitudes are considered, these IR divergences cancel and $\mathcal{M}\left(o\text{-}Ps\to\gamma\gamma\gamma\right)$ is finite. To understand this, remember that an IR divergence in $\mathcal{M}\left(e^+e^-\to\gamma\gamma\gamma\right)$ would be cancelled by IR divergences in radiative corrections to $\mathcal{M}\left(e^+e^-\to\gamma\gamma\right)$, which cannot contribute to $\mathcal{M}\left(o\text{-}Ps\to\gamma\gamma\gamma\right)$.

Nevertheless, and contrary to what is often believed, selection rules do not suffice to enforce analyticity for the standard decay amplitude, because $\mathcal{M}\left(o\text{-}Ps \rightarrow \gamma\gamma\gamma\right)$ must vanish and not simply be finite.

Looking back at the loop model, equivalent to the Bethe-Salpeter loop, a possible solution immediately emerges. Since only off-shell fermions can circulate in the loop, no analytical problem should occur. Taking into account all the cuts, both the oblique and the vertical ones, the correct SPL should be restored. In fact, this is *wrong*, because the loop model amplitude is not gauge invariant for $o\text{-}Ps$ decays, and Low's theorem heavily relies on gauge invariance, i.e. Ward identities. Let us discuss this issue of gauge invariance more precisely.

The source of the problem is in the momentum-dependence of the form factor. One can show that the loop model amplitude is gauge invariant if and only if the form factor accommodates linear shifts $F_B\left(q\right) = F_B\left(q + l_1\right)$, and similarly for l_2, l_3. For a constant form factor, the loop is gauge invariant, and no problem of SPL occurs (the loop amplitude is then obtained from the light-by-light scattering amplitude, for which it is well-known that gauge invariance guarantees correct SPL [20]).

Equivalently, we can trace the problem down to the bremsstrahlung amplitudes. To visualize the situation, let us change the momentum parametrization and draw amplitudes for $o\text{-}Ps \rightarrow e^+\left(p_1\right) e^-\left(p_2\right) \gamma\left(l_1\right)$, as shown in figure 5,

$$\begin{aligned}\mathcal{M}_{IB;1}^{\mu} &= ieF_B\left(p_1 + l_1, p_2\right)\left\{\overline{u}\left(p_1\right)\frac{2p_1^{\mu} + \gamma^{\mu}\not{l}_1}{2p_1\cdot l_1}\gamma^{\alpha}v\left(p_2\right)\right\}e_{\alpha}\left(P\right) \\ &\quad + ieF_B\left(p_1, p_2 + l_1\right)\left\{\overline{u}\left(p_1\right)\gamma^{\alpha}\frac{-2p_2^{\mu} - \not{l}_1\gamma^{\mu}}{2p_2\cdot l_1}v\left(p_2\right)\right\}e_{\alpha}\left(P\right) \qquad (25)\end{aligned}$$

Obviously, when contracted by l_1^{μ}, the two amplitudes fail to cancel each other, due to the different momentum dependences of F_B:

$$l_{1,\mu}\mathcal{M}_{IB;1}^{\mu} = ie\left[F_B\left(p_1 + l_1, p_2\right) - F_B\left(p_1, p_2 + l_1\right)\right]\left\{\overline{u}\left(p_1\right)\gamma^{\alpha}v\left(p_2\right)\right\}e_{\alpha}\left(P\right) \neq 0$$

Thus gauge invariance is violated by the bremsstrahlung parts of the oblique cuts.

Dispersion theory permits the precise identification of the bremsstrahlung processes as responsible for both analyticity and gauge invariance violations. The idea is now to add new structure-dependent (SD) contributions, such that gauge invariance is restored. What we will show is that these new SD contributions also automatically restore the analytical properties for the whole amplitude.

It could seem that the loop model has not achieved much. Worse, by considering it, we have not solved the analytical problems, and we have brought in new gauge invariance problems (since vertical cuts are separately gauge invariant). One could therefore think that it would be much easier to try to find new structure-dependent gauge invariant contributions to cure the standard amplitudes (23), and forget about loops.

Our point of view is that one must consider the loop model as the starting point, being strictly equivalent to the Bethe-Salpeter approaches (16). Another argument is that oblique cuts do a great deal of the work: for a constant form factor, they succeed at curing vertical cut analytical problems. The new SD contributions are therefore accounting for the non-constancy of the form factor. This is far more satisfactory. The loop represents the decay dynamics, and "nearly" succeeds at restoring SPL. The remaining violation comes from the bound state dynamics (non-constant form factor), and is cured by new bound state structure-dependent processes.

Another advantage of the loop approach is that pseudoscalar para-state decays do not necessitate structure-dependent terms. As we will see, thanks to the appearance of a pseudoscalar coupling γ_5, the loop amplitude is gauge invariant, and well-behaved in the SPL. We will explicitly check that for the decay paradimuonium $(\mu^+\mu^-)$ into γe^+e^-, correct SPL is restored through the interference of oblique cuts with the standard approach vertical cuts amplitudes. This analysis is done in the last section, along with the analogous decay $K_S \to \gamma e^+e^-$.

2.2.3 Structure-dependent terms

The violation of gauge invariance by bremsstrahlung-type amplitude is a very well-known phenomenon in pion and kaon physics. One remembers that for a decay process like $\pi^+ (K^+) \to e^+\nu\gamma$, the so-called Inner-Bremsstrahlung (IB) amplitudes combine with Structure Dependent (SD) amplitudes to restore gauge invariance (see for example [28], or [30] for a modern Chiral Perturbation Theory point of view, see also [34] for measurements of structure dependent form factors, and recent literature cited there).

What we here propose is a similar mechanism for positronium. On general grounds, it was foreseeable that *some interplay between the bound state dynamics and the decay process should appear, both ultimately involving photons.* The present model allows us to give some constraints for the general form of those structure terms, exactly like for pseudoscalar pion decay. However, to obtain their exact forms is a very difficult problem.

To analyze the restoration of both the correct SPL behavior and gauge invariance thanks to SD terms, we invoke Low's theorem in its original form (see [18], [28]):

> For a scalar decay process $A^0 \to e^+(p_1)\, e^-(p_2)\, \gamma(k)$ with both structure and bremsstrahlung contributions, the complete amplitude behaves in the SPL as
>
> $$\begin{aligned}\mathcal{M}^\mu\left(A^0 \to e^+e^-\gamma\right) &= F_{Ae^+e^-}(p_1,p_2)\left\{\overline{u}(p_1)\frac{2p_1^\mu+\gamma^\mu \not{k}}{2p_1\cdot k}v(p_2)\right\} \\ &\quad -F_{Ae^+e^-}(p_1,p_2)\left\{\overline{u}(p_1)\frac{2p_2^\mu+\not{k}\gamma^\mu}{2p_2\cdot k}v(p_2)\right\}+\mathcal{O}\left(k^0\right)\end{aligned} \tag{26}$$
>
> if the Ward identity $k_\mu \mathcal{M}^\mu = 0$ holds. Three important consequences:
>
> 1– There is no constant term, i.e. $\mathcal{O}\left(\left(k^0\right)^0\right)$, arising from the form factor.
>
> 2– The form factor appears as evaluated at the same point.
>
> 3– When the process involves only neutral bosons, $\mathcal{M} \sim \mathcal{O}\left(k^0\right)$.

To understand the content of the first point, let us expand $F_B(p_1+l_1,p_2)$ in (25) around $l_1^0 = 0$. The expansion of the form factor will bring a new constant term

$$\begin{aligned}F_B(p_1+l_1,p_2)\frac{2p_1^\mu+\gamma^\mu \not{l}_1}{2p_1\cdot l_1} \quad &\to \quad F_B(p_1,p_2)\frac{2p_1^\mu+\gamma^\mu \not{l}_1}{2p_1\cdot l_1} \\ &\qquad +2ap_1^\mu\frac{\partial F_B(p_1,p_2)}{\partial p_1}+\mathcal{O}\left(l_1^0\right)\end{aligned}$$

for some a. It is this additional constant term that must be cancelled by SD amplitudes to preserve gauge invariance, hence leading to Low's theorem (26).

The second point can be rephrased as follows: near $k^0 = 0$, the complete amplitude behaves exactly as if the form factor was a constant (i.e. it is evaluated at the symmetric point (24) for all contributions). This is the key. The result is neatly expressed as

In the soft-photon limits, the loop amplitudes behave exactly like the constant form factor loop model, because structure-dependent amplitudes interfere with oblique cuts to cancel their dependences on the form factor variations. As a result, positronium decay amplitudes vanish in the SPL, since for a constant form factor loop model, oblique cuts interfere destructively with vertical cuts.

This result is quite powerful, because the dynamics of each contribution is manifest. This opens the way to an evaluation of their respective scalings in α and γ, as we now discuss.

2.2.4 Scaling and NRQED

At first sight, all contributions (vertical, oblique and structure) should be of comparable numerical size, all of them being of the same order in α. This is in fact not at all the case. A very special feature of electromagnetic (loosely) bound states is that the binding energy can be expressed in terms of α

$$\gamma \approx m\alpha/2$$

It is one of the great virtue of NRQED to provide a consistent framework to re-order the perturbation series according to powers of $\alpha^n\gamma^m \to \alpha^{n+m}$.

Another virtue of NRQED is that the starting point is the Bethe-Salpeter loop model (16), i.e. the same loop model as (17). This would mean that all the cuts should somehow be contained in NRQED calculations. If it is true that lowest order NRQED starts by setting $q_0 = 0$ in (16) (because kinetic energy is subleading compared to momentum), systematic corrections to this approximation are then calculated, and the four-dimensional loop is "reconstructed". In other words, the lowest order NRQED result is our vertical cut results, while oblique cuts correspond to higher order corrections.

Our approach could then be viewed as complementary to NRQED perturbative expansions. Dispersion relations put forward basic principles, and should provide interesting constraints on the systematics of NRQED.

Structure-dependent contributions are not contained in NRQED. Their form is unknown and so is their numerical importance. Their existence means that the issue of gauge invariance for the NRQED expansion is irrelevant, because hopeless. This means that the "reconstruction" of the four-dimensional BS loop is not constrained by gauge invariance. However, by showing that structure-terms are contributing at the order α^3, one proves that the spurious gauge dependence and violation of analyticity of NRQED is also of that order, hence validating the present $\mathcal{O}\left(\alpha^2\right)$ theoretical predictions for the rates.

Let us review the different contributions. What follows is not a rigorous demonstration, but rather a qualitative analysis. Nevertheless, the exact evaluation should not alter much our conclusions.

i- The vertical cuts alone are sufficient to reproduce the known lowest order decay rate [7]

$$\Gamma\left(o\text{-}Ps \to \gamma\gamma\gamma\right) = \frac{2\left(\pi^2-9\right)}{9\pi}\alpha^6 m$$

This result is obtained in the static limit, i.e. with $F_B \propto \delta^{(3)}\left(\mathbf{q}\right)$. Let us also give the differential rate [11],

$$\frac{d\Gamma\left(o\text{-}Ps \to \gamma\gamma\gamma\right)}{dx_1} = \frac{2\alpha^6 m}{9\pi}\left[\tfrac{2(2-x_1)}{x_1} + \tfrac{2(1-x_1)x_1}{(2-x_1)^2} + \left[\tfrac{4(1-x_1)}{x_1^2} - \tfrac{4(1-x_1)^2}{(2-x_1)^3}\right]\ln\left(1-x_1\right)\right] \tag{27}$$

with $x_1 = l_1^0/2M$ the reduced photon energy. The spectrum (27) illustrates the violation of analyticity. Near zero, it vanishes linearly

$$\frac{d\Gamma\left(o\text{-}Ps\rightarrow\gamma\gamma\gamma\right)}{dx_1}\overset{x_1\sim 0}{\sim}\frac{5}{3}x_1 \tag{28}$$

while we would expect a x_1^3 behavior instead. Indeed, the phase space alone is

$$\frac{d\Gamma\left(o\text{-}Ps\rightarrow\gamma\gamma\gamma\right)}{dx}\sim\frac{2\alpha^6 m}{9\pi}\left[2x_1\right] \tag{29}$$

Since the amplitude should behave as x_1 near $x_1 = 0$, the differential rate should exhibit a x_1^3 behavior in the SPL. As we have seen, the additional contributions restore the correct SPL.

ii- The oblique cuts are suppressed by three effects: they disappear in the limit of vanishing binding energy ($\gamma\rightarrow 0$), the dispersion relation integration range is reduced compared to the one for vertical cuts, and the three-photon phase-space enhances vertical cuts against oblique ones.

The first point is obvious: if the e^+e^- pair emerges on-shell from the bound state, there is no room for oblique cuts. In fact, this limit is pathological, because oblique cuts are still there, but contribute only when one of the photon energies is strictly zero, to enforce SPL. Anyway, their complete contribution should somehow be proportional to the binding energy γ, i.e. the mass default of the e^+e^- inside the positronium.

To explain the second point, consider the dispersion integral (19b). The lower bound $s_{\min(l_1)}$ is increasing as l_1^0 increases. Indeed, if the photon is carrying away some energy, the initial bound state mass squared s must be greater to create on-shell intermediate e^+e^-. Therefore, oblique cuts should be decreasing functions of the photon energies.

The first two effects combine to suppress oblique cut contributions everywhere except near zero photon energy. Therefore, oblique cut contributions to the decay rate will be further (substantially) suppressed by the phase-space (27), peaked for high photon energies.

Due to all those effects, one could state that oblique cuts should contribute at the order $\gamma^2\approx\alpha^2$, i.e. $\mathcal{D}\approx\alpha^2\mathcal{V}$. Since the argument is qualitative, it could happen that they are enhanced for some reason (or further suppressed), but we do not see any compelling evidence of such pathologies.

iii- Finally, **structure terms** can be estimated by taking the conservative view that their importance is just what is required to preserve gauge invariance and SPL. Since they compete with oblique cuts, they are of the same magnitude, i.e. $\mathcal{O}\left(\alpha^2\right)$. However, structure terms account for the *variation* of the form factor. For a Schrödinger-like form factor

$$\mathcal{F}\left(\mathbf{q}^2\right)=\frac{8\pi\gamma}{\left(\mathbf{q}^2+\gamma^2\right)^2}$$

The variations will be a much more peaked function of $\mathbf{q}^2$, leading to a smaller result for the dispersion integral (19c). All in all, we should gain a power of the binding energy $\mathcal{S}\approx\gamma\mathcal{D}$. In terms of the fine-structure constant, structure term should be of order α^3, i.e. $\mathcal{S}\approx\alpha^3\mathcal{V}$.

Of course, this is not very convincing. If the pion decay can be used as a guide, structure terms should indeed be highly suppressed. However, the very small mass gap between the parapositronium and orthopositronium could lead to huge enhancements, if one imagines structure term dynamics as driven by intermediate virtual parapositronium.

In conclusion, we argue that, provided that no pathological effects come into play, oblique cuts could be considered as small $\mathcal{O}\left(\alpha^2\right)$ corrections, and structure dependent terms neglected at the present theoretical precision level. Anyway, no answer to the orthopositronium lifetime puzzle could be viewed as entirely satisfactory as long as no definite estimation or calculation of structure-dependent processes has been achieved.

3 Form Factor and Parapositronium Decay

We have established the correspondence between (10) and (15). Let us construct an alternative, but equivalent, dispersion procedure specific to the two-photon case that will be used in explicit calculations.

By computing the trace in (10), the tensor structure factorizes and we are left with

$$\mathcal{M}\left(p\text{-}Ps \rightarrow \gamma\gamma\right) = 8me^2\varepsilon^{\mu\nu\rho\sigma}l_{1,\rho}l_{2,\sigma}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\mathcal{I}\left(M^2\right) \tag{30}$$

and the decay rate is expressed simply as

$$\Gamma\left(p\text{-}Ps \rightarrow \gamma\gamma\right) = 16\pi\alpha^2m^2M^3\left|\mathcal{I}\left(M^2\right)\right|^2 \tag{31}$$

In this equation, $\mathcal{I}\left(M^2\right)$ can be viewed as an effective form factor, modeled as the electron-positron loop with the coupling F_B. There is only one term in $\mathcal{I}\left(M^2\right)$ since the direct and crossed amplitudes are equal under $q \rightarrow -q$, i.e. an allowed variable change as $F_B\left(q^2, P\cdot q\right) = F_B\left(q^2, -P\cdot q\right)$, and we write

$$\mathcal{I}\left(P^2\right) = \eta\int\frac{d^4q}{\left(2\pi\right)^4}F_B\frac{1}{\left(q-\frac{1}{2}P\right)^2-m^2}\frac{1}{\left(q+\frac{1}{2}P\right)^2-m^2}\frac{1}{\left(q-\frac{1}{2}P+l_1\right)^2-m^2} \tag{32}$$

It is to evaluate the effective form factor $\mathcal{I}\left(P^2\right)$ that we will now use dispersion techniques. Remark that the factorization of the tensor part is interesting, since gauge invariance is manifest, and that $\mathcal{I}\left(P^2\right)$ is convergent while the amplitude (10) is superficially divergent.

The factor η is introduced because there is a subtlety in the above factorization. Indeed, there is an arbitrariness in the choice of variable for the dispersion integral. This situation is well-known for the photon vacuum polarization:

$$\Pi^{\mu\nu}\left(k^2\right) = \left(k^2g^{\mu\nu}-k^\mu k^\nu\right)\Pi_1\left(k^2\right) = \left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)\Pi_2\left(k^2\right) \tag{33}$$

where one writes a dispersion relation for $\Pi_1\left(k^2\right)$, which is less divergent than $\Pi_2\left(k^2\right)$ due to the factorization of the tensor structure. The analogue of (33) for our case is

$$\mathcal{M}\left(p\text{-}Ps \rightarrow \gamma\gamma\right) = 8me^2\varepsilon^{\mu\nu\rho\sigma}l_{1,\rho}l_{2,\sigma}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\mathcal{I}_1\left(M^2\right) = 8me^2\varepsilon^{\mu\nu\rho\sigma}\frac{l_{1,\rho}}{M}\frac{l_{2,\sigma}}{M}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\mathcal{I}_2\left(M^2\right)$$

We will choose to write a dispersion relation for $\mathcal{I}_2\left(M^2\right)$, and this corresponds to the choice $\eta = P^2/M^2$. This seemingly arbitrary choice is in fact necessary to be consistent with the general development, see (13), (14). In other words, the dispersion relation was built on the whole amplitude, so it is clear that the photon momenta appearing in the tensor structure were also incorporated, i.e. they were reduced as $l_{1,\rho} \rightarrow \overline{l_{1,\rho}} \times \sqrt{P^2}/2$. That is the reason why we must include a factor P^2 into the effective form factor $\mathcal{I}\left(P^2\right)$.

Let us give a general expression for $\mathcal{I}\left(P^2\right)$ as a dispersion integral. The loop integral $\mathcal{I}\left(P^2\right)$ has an imaginary part obtained by cutting the propagators

$$\operatorname{Im}\mathcal{I}\left(P^2\right)=\frac{1}{2}\frac{P^2}{M^2}\int\frac{d^4q}{\left(2\pi\right)^4}F_B\frac{2\pi i\delta\left(\left(q-\frac{1}{2}P\right)^2-m^2\right)2\pi i\delta\left(\left(q+\frac{1}{2}P\right)^2-m^2\right)}{\left(q-\frac{1}{2}P+l_1\right)^2-m^2}$$

Proceeding exactly like in the general case (see appendix), using an unsubstracted DR, we reach the two equivalent forms

$$\mathcal{I}\left(M^2\right) = \frac{C\phi_o}{2M^2}\frac{1}{32\pi^2}\int_{4m^2}^{+\infty}ds\mathcal{F}\left(s/4-m^2\right)\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{s}}}{1-\sqrt{1-\frac{4m^2}{s}}}\right] \tag{34a}$$

$$= \frac{C\phi_o}{2M^2}\int\frac{d^3\mathbf{q}}{\left(2\pi\right)^3}\mathcal{F}\left(\mathbf{q}^2\right)\frac{1}{\sqrt{\left|\mathbf{q}\right|^2+m^2}+\left|\mathbf{q}\right|\cos\theta_{\mathbf{q}}} \tag{34b}$$

where we have set $F_B\equiv C\phi_o\mathcal{F}\left(\mathbf{q}^2\right)\left(\mathbf{q}^2+\gamma^2\right)$ (see 12). The first expression is obtained by integration over the angular variable $\theta_{\mathbf{q}}$.

This is our third representation for the same decay amplitude : the first is the loop integral (10), the second is the well-known amplitude (15) (or (8), with no approximation for projectors) viewed as a dispersion integral for the amplitude, and the third is the present dispersion integral (34a,b) for the effective loop form factor $\mathcal{I}\left(M^2\right)$. All three procedures are strictly equivalent to each other.

3.1 Decay Rate in the Static Limit

Let us first analyze the static limit, i.e. $\gamma^2\to 0$ for the form factor. To compute the decay rate in that limit, we do not need to specify F_B yet. We just need to know that the function $\mathcal{F}\left(\mathbf{k}^2\right)$ is normalized to unity and behaves as a delta function of the momentum in the limit of vanishing binding energy:

$$\int\frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\mathcal{F}\left(\mathbf{k}^2\right)=1,\qquad\lim_{\gamma\to 0}\mathcal{F}\left(\mathbf{k}^2\right)=\left(2\pi\right)^3\delta^{(3)}\left(\mathbf{k}\right) \tag{35}$$

By setting $\mathcal{F}\left(\mathbf{q}^2\right)=\left(2\pi\right)^3\delta^{(3)}\left(\mathbf{q}\right)$ in (32), we must recover exactly the lowest order decay rate

$$\Gamma\left(p\text{-}Ps\to\gamma\gamma\right)=\frac{1}{2}\alpha^5 m \tag{36}$$

In some sense, the static limit serves as a boundary condition for the behavior of $\mathcal{F}$.

Using (34b), we get

$$\mathcal{I}\left(M^2\right)=\frac{C\phi_o}{2M^2}\int\frac{d^3\mathbf{q}}{\left(2\pi\right)^3}\left(2\pi\right)^3\delta^{(3)}\left(\mathbf{q}\right)\frac{1}{\sqrt{\left|\mathbf{q}\right|^2+m^2}+\left|\mathbf{q}\right|\cos\theta}=\frac{C\phi_o}{2M^2}\left[\frac{1}{m}\right]$$

Importantly, this result is independent of the binding energy : the loop do not introduce any corrections in the static limit. The decay rate in that limit is therefore:

$$\Gamma\left(p\text{-}Ps\to\gamma\gamma\right)=\frac{1}{2}\alpha^5 m\left(\frac{m^2}{M}C^2\right) \tag{37}$$

with $\left|\phi_0\right|^2=\alpha^3m^3/8\pi$. It remains to match C such that purely kinematic corrections vanish (M factors in the above formula arise from products like $\left(l_1\cdot l_2\right)=$

$M^2/2$ and from the $1/2M$ decay width factor, while m comes from electron propagators in the loop and from the wavefunction ϕ_0). With the definition $C = \sqrt{M}/m$, the decay rate is exactly $\frac{1}{2}\alpha^5 m$ as it should. In other words, the value for C obtained by matching (15) and (8) is such that no correction arises from factors M/m in the static limit.

To conclude, let us repeat that we have not specified the form factor. This means that any form factor which has a three-dimensional delta function limit for $\gamma^2 \to 0$ gives the correct lowest order decay rate $\frac{1}{2}\alpha^5 m$. In the following, we shall present three forms, all built on the Schrödinger momentum wavefunction.

3.2 Schrödinger Form Factors and Binding Energy Corrections

The formula (34a) is a very simple expression: the decay rate (31) for any form factor is obtained by a one-dimensional integral. This is a very nice calculational tool that we will now use. We will go through different calculations of $\Gamma\left(p\text{-}Ps \to \gamma\gamma\right)$, obtained for specific choices of F_B (or equivalently, $\mathcal{F}\left(\mathbf{q}^2\right)$). Namely:

$$\mathcal{F}_I\left(\mathbf{q}^2\right) = \frac{8\pi\gamma}{\left(\mathbf{q}^2+\gamma^2\right)^2} \tag{38a}$$

$$\mathcal{F}_{II}\left(\mathbf{q}^2\right) = \frac{32\pi\gamma^3}{\left(\mathbf{q}^2+\gamma^2\right)^3} \tag{38b}$$

$$\mathcal{F}_{III}\left(\mathbf{q}^2\right) = \frac{2\gamma}{|\mathbf{q}|}\arctan\frac{|\mathbf{q}|}{\gamma} \times \mathcal{F}_I\left(\mathbf{q}^2\right) \tag{38c}$$

where $\gamma^2 = m^2 - M^2/4 \approx m^2\alpha^2/4$ related to the binding energy $E_B = M - 2m = -m\alpha^2/4$. All these form factors satisfy the delta limit property (35), but are differently peaked near $|\mathbf{q}| = 0$ (see figure 7). As a result, we expect that all of them will reproduce the lowest order decay rate (36), but will introduce some γ-dependent corrections proportional to their spreading around $|\mathbf{q}| = 0$. Such corrections can then be expressed in terms of the fine-structure constant, yielding a corrected lowest order rate of

$$\Gamma\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{1}{2}\alpha^5 m\left(1 + \delta_\Gamma\left(\mathcal{F}\right)\right)$$

The physics of each form factor is discussed below. For now, let us use the formula (34a) to get the effective form factors and decay rates corrections $\delta_\Gamma\left(\mathcal{F}_{I,II,III}\right)$. For the first one, we find

$$\mathcal{I}_I\left(M^2\right) = \frac{C\phi_o}{M^3}\frac{2}{\pi}\arctan\frac{M}{2\gamma} \tag{39}$$

The integral needed for this calculation is (see [21])

$$\int_0^1 \frac{dx}{x_o - x}\ \ln\left[\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}\right] \overset{x_o \geq 1}{=} 2\arctan^2\frac{1}{\sqrt{x_o - 1}} \tag{40}$$

and its derivative $\partial/\partial x_o$. The decay rate is

$$\Gamma_I\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{1}{2}\alpha^5 m\left(\frac{4m^2}{M^2}\right)\left(\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right)^2 \tag{41}$$

where we used $|\phi_0|^2 = \alpha^3 m^3/8\pi$ and $C = \sqrt{M}/m$. By expanding this result around $\gamma = 0$, and expressing corrections as a series in α, we recover the standard result as zeroth order

$$\Gamma_I\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{1}{2}\alpha^5 m \left(1 - \frac{\alpha}{\pi} + \frac{1}{8}\alpha^2 + \mathcal{O}\left(\alpha^3\right)\right)^2$$
$$\approx \frac{1}{2}\alpha^5 m \left(1 - 0.637\alpha + 0.351\alpha^2 + \mathcal{O}\left(\alpha^3\right)\right)$$

Proceeding similarly for the other two, using (40) for $\mathcal{F}_{II}$, and numerical integration for $\mathcal{F}_{III}$, we find

$$\Gamma_{II}\left(p\text{-}Ps \to \gamma\gamma\right) \approx \frac{1}{2}\alpha^5 m \left(1 - 0.25\alpha^2\right)$$
$$\Gamma_{III}\left(p\text{-}Ps \to \gamma\gamma\right) \approx \frac{1}{2}\alpha^5 m \left(1 - 1.73\alpha^2\right)$$

If we naively combine the present corrections with radiative corrections up to order $\alpha^2 \ln \alpha$:

$$\Gamma_{p\text{-}Ps} = \frac{\alpha^5 m}{2}\left(1 - \left(5 - \frac{\pi^2}{4}\right)\frac{\alpha}{\pi} + 2\alpha^2 \ln\frac{1}{\alpha} + \delta_\Gamma\left(\mathcal{F}\right)\right)$$

the theoretical value is modified as

Lowest-order Form factor	Corrections δ_Γ	Decay Rate, with Radiative Corrections and δ_Γ
Static Limit	0	$7.9895 \times 10^9\,\mathrm{sec}^{-1}$
$\mathcal{F}_I$	-0.637α	$7.9527 \times 10^9\,\mathrm{sec}^{-1}$
$\mathcal{F}_{II}$	$-0.25\alpha^2$	$7.9894 \times 10^9\,\mathrm{sec}^{-1}$
$\mathcal{F}_{III}$	$-1.73\alpha^2$	$7.9887 \times 10^9\,\mathrm{sec}^{-1}$
Experiments		$(7.9909 \pm 0.0017) \times 10^9\,\mathrm{sec}^{-1}$

where the experimental measurement (1) is also included for comparison.

3.2.1 Discussion

In view of the above table, it appears that the first form factor $\mathcal{F}_I$ leads to a too small value for $\Gamma\left(p\text{-}Ps \to \gamma\gamma\right)$, since it introduces new order α corrections. The problem is in the form factor, which does not converge fast enough towards the static limit delta. In other words, for a given γ^2, $\mathcal{F}_I\left(\mathbf{q}^2\right)$ is not enough peaked around $|\mathbf{q}| = 0$. On the other hand, the more peaked $\mathcal{F}_{II}$ and $\mathcal{F}_{III}$ form factors lead to acceptable corrections.

It is time now to consider the physical content of each form factor. In summary, the $\mathcal{F}_I$ is just the Shrödinger momentum wavefunction for the bound state, the $\mathcal{F}_{II}$ form factor has no clear signification and only serves the purpose of illustration, while $\mathcal{F}_{III}$ can be viewed as a Coulomb binding corrected Schrödinger form factor, in the spirit of the Sommerfeld factor. Of course, one could argue that the form factor to be used must be linked somehow to the Bethe-Salpeter wavefunction. In fact, it is the first form factor $\mathcal{F}_I$ that emerges from the reduction of the Barbieri-Remiddi wavefunction through dispersion relations (15). Despite of that, we will argue that it is the $\mathcal{F}_{III}$ form factor that must be considered if one is interested in $\mathcal{O}\left(\alpha\right)$ accuracy (this discussion is much inspired from [11]).

The fact that the exact integration of the $\mathcal{F}_I$ form factor leads to excessive order α corrections $\delta_\Gamma\left(\mathcal{F}_I\right)$ is not new. Let us stress first that those corrections are γ-dependent (see (41)), i.e. *it is a binding energy effect*. Such effects, being non-perturbative, are quite difficult to be dealt with. Even in the $\mathcal{O}\left(\alpha\right)$ calculation of [11], much space is devoted to the discussion of such effects. Double-countings plague bound state calculations, and it is quite possible that the present excessive

$\mathcal{O}(\alpha)$ corrections should be discarded because they are already taken into account in $\mathcal{O}(\alpha)$ calculations [13].

Indeed, the standard approach to these excessive corrections is simply to discard them. Either they are assumed to be part of higher order radiative corrections to the scattering process ([10], [11]), or alternatively, they are considered as part of the relativistic corrections to the bound state wavefunction [13], [14]. In any case, they are interpreted as accounting for some modification of the intermediate e^+e^- state: either as emitted from the bound state, either as entering the scattering process.

This dual view is further enhanced by the following: those additional corrections disappear in the static limit, i.e. when $\gamma^2 \to 0$, so they appear as originating from the bound state dynamics. Alternatively, looking back at (34b), it appears that the electron propagator between the two photons plays a crucial role in introducing the additional corrections. If it is frozen (i.e. replaced by $1/m$), the corrections disappear. In other words, one could view the decay process instead of the bound state dynamics as responsible for the excessive corrections.

From the point of view of the BS equation, things are quite clear. The Schrödinger momentum wavefunction is built as the Coulomb photon exchange ladder approximation (see figure 8a). Relativistic corrections to the wavefunction must be considered to account for the covariant photon exchange. This corresponds to the relativistic corrections point of view. The radiative corrections view arises by considering the binding graph (figure 8b). As shown for example in [11], this $\mathcal{O}(\alpha)$ radiative correction for the scattering amplitude reproduces the lowest order decay amplitude, because its Coulomb photon exchange part has already been accounted for in the construction of the wavefunction.

What we propose is to view the third form factor as a representation of the modification of the intermediate e^+e^- state. When dispersion relations are used, one consider intermediate states as asymptotic states, and integrate over the corresponding phase-space. This is the application of the optical theorem for absorptive parts. Let us assume that, due to the modification of the asymptotic Hilbert space by the long-range Coulomb interactions, a correction factor must be introduced in

$$\mathcal{M}\left(p\text{-}Ps \to \gamma\gamma\right) \sim \int d^3\mathbf{q}\, \psi\left(\mathbf{q}^2\right) \times S\left(\mathbf{q}\right) \times \mathcal{M}\left(e^-_{\mathbf{q}} e^+_{-\mathbf{q}} \to \gamma\gamma\right) \tag{42}$$

and that this factor is

$$S\left(\mathbf{q}\right) = \frac{2\gamma}{|\mathbf{q}|} \arctan \frac{|\mathbf{q}|}{\gamma} \tag{43}$$

In some sense, one could think of $S\left(\mathbf{q}\right)$ as the Sommerfeld factor of Harris and Brown [8]. Due to long range Coulomb interactions, the wavefunction at contact ϕ_o is "renormalized" by

$$|\phi_o|^2 \to |\phi_o|^2 \frac{2\pi\alpha/v}{1 - e^{-2\pi\alpha/v}} \approx |\phi_o|^2 \left(1 + \frac{\pi\alpha}{v}\right)$$

by using $v >> \alpha$, i.e. $|\mathbf{q}| >> \gamma$, with v the center-of-mass velocity. Correspondingly, $S\left(\mathbf{q}\right)$ for small binding energy is

$$S\left(\mathbf{q}\right) \approx \frac{\pi\gamma}{|\mathbf{q}|} \approx \frac{\pi\alpha}{2v}$$

The factor of 2 arises because $S\left(\mathbf{q}\right)$ is defined at the amplitude level, while the Sommerfeld factor is at the decay rate level.

The same arctangent factor arises in the work of [10], [11] on $\mathcal{O}(\alpha)$ radiative corrections, by a detailed analysis of the binding graph and its Coulombic part (which is, after all, an initial state Coulombic interaction).

In conclusion, if one is interested in the lowest approximation (i.e. the $m\alpha^5/2$) the static limit is just fine. If one wants the lowest approximation to $\mathcal{O}(\alpha)$ accuracy, one must consider it as arising from the $\mathcal{O}(\alpha)$ binding graph, as demonstrated by Adkins [11]. In other words, by taking *only* $\mathcal{O}(\alpha)$ diagrams for the scattering amplitude $e^+e^- \to \gamma\gamma$, one finds

$$\Gamma\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{1}{2}m\alpha^5\left(1 - \frac{\alpha}{\pi}\left(5 - \frac{\pi^2}{4}\right)\right) + \mathcal{O}\left(\alpha^2\right)$$

The 1 in the above expression is the lowest approximation to $\mathcal{O}(\alpha)$ accuracy, arising from the $\mathcal{O}(\alpha)$ amplitude. To reproduce it using our effective form factor method, the Coulomb corrected Schrödinger form factor $\mathcal{F}_{III}$ is to be used, as in [11].

What we gained using our method is a better understanding of its link with the Sommerfeld factor, i.e. asymptotic Coulomb interactions. $\mathcal{F}_{III}$ being more peaked than $\mathcal{F}_I$, the suppression of the intermediate phase-space is interpreted as a manifestation of the iterative structure of Bethe-Salpeter construction of the wavefunction.

Therefore, the method of [11] is nothing but the rephrasing of that of Harris and Brown [8]. However, the method of [11] or the present integration of $\mathcal{F}_{III}$ is surely more appropriate, because one does not rely on static limits, hence avoids the well-known static divergence $1/v$ of radiative corrections to $e^-e^+ \to \gamma\gamma$ [26].

3.2.2 Improving the perturbation series

It is interesting to reverse the previous argument. Instead of curing the form factor so that one avoids double-counting, it would be more adequate to modify radiative corrections, keeping the $\mathcal{F}_I$ form factor. Indeed, the form factor accounts for most of the $\mathcal{O}(\alpha)$ corrections. We found

$$\Gamma_I\left(p\text{-}Ps \to \gamma\gamma\right) = \frac{\alpha^5 m}{2}\left(1 - 2\frac{\alpha}{\pi} + \mathcal{O}\left(\alpha^2\right)\right) \approx \frac{\alpha^5 m}{2}\left(1 - 0.637\alpha + \mathcal{O}\left(\alpha^2\right)\right)$$

while complete corrections are (4a)

$$\Gamma_{p\text{-}Ps} = \frac{\alpha^5 m}{2}\left(1 - \left(5 - \frac{\pi^2}{4}\right)\frac{\alpha}{\pi} + \mathcal{O}\left(\alpha^2\right)\right) \approx \frac{\alpha^5 m}{2}\left(1 - 0.806\alpha + \mathcal{O}\left(\alpha^2\right)\right)$$

In other words, the decay rate can be expressed as

$$\begin{aligned}\Gamma_{p\text{-}Ps} &= \frac{\alpha^5 m}{2}\frac{4m^2}{M^2}\left(\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right)^2\left(1 - \left(3 - \frac{\pi^2}{4}\right)\frac{\alpha}{\pi} + \mathcal{O}\left(\alpha^2\right)\right) \\ &\approx \frac{\alpha^5 m}{2}\frac{4m^2}{M^2}\left(\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right)^2\left(1 - 0.170\alpha + \mathcal{O}\left(\alpha^2\right)\right) \qquad (44)\end{aligned}$$

This last form is very interesting because binding energy corrections (i.e. γ-dependent) are singled out, while the resulting purely radiative corrections are much reduced. This means that one could, at least in principle, express the decay rate as non-perturbative binding energy corrections times a rapidly converging perturbation series of radiative corrections. This is exactly what we have achieved to order α.

It is worthwhile to note that (44) also accounts for most of the $\mathcal{O}\left(\alpha^2\right)$ corrections. Omitting logarithmic corrections, the decay rate is given by

$$\begin{aligned}\Gamma_{p\text{-}Ps} &\approx \frac{\alpha^5 m}{2}\left(1 - 0.806\alpha + 0.52\alpha^2 + \mathcal{O}\left(\alpha^3\right)\right) \\ &\approx \frac{\alpha^5 m}{2}\frac{4m^2}{M^2}\left(\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right)^2\left(1 - 0.170\alpha + 0.06\alpha^2 + \mathcal{O}\left(\alpha^3\right)\right) \qquad (45)\end{aligned}$$

This appears as coincidental (especially in view of the many interferences occurring among $\mathcal{O}\left(\alpha^2\right)$ corrections, see [12]), but it is nevertheless intriguing. It may be the signal that an improved basis can be built for the positronium perturbative calculation, if one avoids NRQED systematic α expansion of binding energy effects. (Remark that this formulation does not cause any problem regarding logarithmic corrections; the modification amount only to a reduction of the coefficient C_p of the $\alpha^3 \ln \alpha$ term to $-3.919\,(1)$ from $-7.919\,(1)$)

Of course, it would be very interesting to build the corresponding expression for orthopositronium. Unfortunately, as extensively discussed, both oblique cuts and (unknown) structure-dependent contributions render the calculation highly challenging.

4 Paradimuonium Decay into $\gamma e^+ e^-$

The present section concerns paradimuonium, the singlet $\mu^+\mu^-$ bound state [27]. This state has not been observed yet. The reason why we consider the decay p-$Dm \rightarrow \gamma e^+ e^-$ is that it is the simplest process where a photon is kinematically allowed to have a vanishing energy (Note that the positronium decay p-$Ps \rightarrow \gamma\nu\overline{\nu}$ via a Z^0 has the same dynamics, but has not been observed either). Therefore, we will be able to illustrate that the loop model has a correct soft-photon behavior in the case of parapositronium or paradimuonium decay.

The decay p-$Dm \rightarrow e^+e^-\gamma$ is shown in figure 9. The simplicity of the process comes from the pseudoscalar nature of the paradimuonium, which allows a manifestly gauge invariant treatment throughout (hence the absence of structure-dependent terms). The amplitude for p-$Dm \rightarrow e^+e^-\gamma$ can be directly obtained from the positronium to two photon decay amplitude (30), (32):

$$\mathcal{M}\left(p\text{-}Dm \rightarrow e^+e^-\gamma\right) = 8me^3\varepsilon^{\mu\nu\rho\sigma}k_\rho t_\sigma \varepsilon^*_\mu\left(k\right)\frac{\left\{\overline{u}\left(p\right)\gamma_\nu v\left(p'\right)\right\}}{t^2+i\varepsilon}\mathcal{I}\left(M^2,x\right) \tag{46}$$

with $x = 2k^0/M$ the reduced photon energy, M the dimuonium mass and m the muon mass. The effective form factor loop integral is given by

$$\mathcal{I}\left(P^2,x\right) = \eta\int\frac{d^4q}{\left(2\pi\right)^4}F_B\frac{1}{\left(q-\frac{1}{2}P\right)^2-m^2}\frac{1}{\left(q+\frac{1}{2}P\right)^2-m^2}\frac{1}{\left(q-\frac{1}{2}P+k\right)^2-m^2}$$

where $\eta = P^2/M^2$ and $t^2 = P^2(1-x)$ (the only difference with p-$Ps \rightarrow \gamma\gamma$ being that $t^2 \neq 0$). Gauge invariance is ensured by the factorized tensor structure, i.e. by the antisymmetric Levi-Civita tensor.

From the decay amplitude, the differential width is obtained as

$$\frac{d\Gamma\left(p\text{-}Dm \rightarrow e^+e^-\gamma\right)}{dx} = \frac{16\alpha^3}{3}m^2\,M^3\left|\mathcal{I}\left(M^2,x\right)\right|^2\rho\left(x,a\right) \tag{47}$$

with the phase space spectrum

$$\rho\left(x,a\right) = \sqrt{1-\frac{a}{1-x}}\left[a+2\left(1-x\right)\right]\frac{x^3}{\left(1-x\right)^2} \tag{48}$$

where $a = 4m_e^2/M^2$, and the bounds on x are $[0, 1-a]$. This spectrum is highly peaked just below $1-a$, as shown on figure 10.

We are going to calculate the integral $\mathcal{I}\left(M^2,x\right)$ using dispersion relations. Because $t^2 \neq 0$ (i.e. $x \neq 1$), two different types of cuts will contribute (figure 11).

We have shown that considering the vertical cuts is strictly equivalent to the decay amplitude calculation done using formula (15) where Γ is now the (amputated) scattering amplitude $\mu^+(k')\,\mu^-(k)\to\gamma\gamma^*\to e^+e^-\gamma$ with on-shell muons.

What we are going to show is that *the amplitude for p-Dm $\to e^+e^-\gamma$ has a correct soft-photon limit only if we take all the cuts into account*. By this we mean that each cut gives a contribution to the amplitude that behaves as a constant when the photon energy goes to zero. The combination of the vertical and oblique cuts, on the contrary, forces the amplitude to vanish in that limit. We thus recover the analytical behaviour expected from Low's theorem [18] for the decay $p\text{-}Dm\to\gamma\gamma^*$ involving only neutral bosons.

4.1 Dispersion Relations and Soft photon limit

In this section, we will analyze properties of the effective form factor $\mathcal{I}\left(P^2,x\right)$. To keep the discussion as general as possible, we will not specify the form factor F_B.

i- The absorptive part is found by cutting the relevant propagators as

$$\begin{aligned}
\operatorname{Im}\mathcal{I}_1\left(P^2,x\right) &= \eta\int\frac{d^4q}{2\left(2\pi\right)^4}F_B\frac{2\pi i\delta\left(\left(q-\frac{1}{2}P\right)^2-m^2\right)2\pi i\delta\left(\left(q+\frac{1}{2}P\right)^2-m^2\right)}{\left(q-\frac{1}{2}P+k\right)^2-m^2}\\
\operatorname{Im}\mathcal{I}_2\left(P^2,x\right) &= \eta\int\frac{d^4q}{2\left(2\pi\right)^4}F_B\frac{2\pi i\delta\left(\left(q+\frac{1}{2}P\right)^2-m^2\right)2\pi i\delta\left(\left(q-\frac{1}{2}P+k\right)^2-m^2\right)}{\left(q-\frac{1}{2}P\right)^2-m^2}
\end{aligned}$$

for the vertical and oblique cuts, respectively. By a straightforward integration, the first expression gives ($s=P^2$):

$$\operatorname{Im}\mathcal{I}_1\left(s,x\right)=\frac{\eta}{s}\frac{1}{16\pi x}F_B\left(q_0=0,|\mathbf{q}|=\sqrt{s/4-m^2}\right)\ln\left[\frac{1+\sqrt{1-4m^2/s}}{1-\sqrt{1-4m^2/s}}\right]\theta\left(s-4m^2\right) \tag{49}$$

while the second one cannot be completely integrated without specifying F_B:

$$\operatorname{Im}\mathcal{I}_2\left(s,x\right)=\frac{\eta}{s}\frac{1}{16\pi x}\int_{q_{\min}}^{q_{\max}}\frac{dq_0}{q_0}F_B\left(q_0,|\mathbf{q}|=\sqrt{q_0^2+q_0\sqrt{s}+s/4-m^2}\right)\theta\left(s-\frac{4m^2}{1-x}\right) \tag{50}$$

with the bounds given by

$$q_{\min}=-\tfrac{x\sqrt{s}}{4}\left(1+\sqrt{1-\tfrac{4m^2}{s(1-x)}}\right),\ q_{\max}=-\tfrac{x\sqrt{s}}{4}\left(1-\sqrt{1-\tfrac{4m^2}{s(1-x)}}\right) \tag{51}$$

Interestingly, the second cuts contribute for $q_0\neq 0$. This is in sharp contrast with the two-photon decay, since there only the first cuts exist. Further, approximated bound state wavefunctions where a $\delta\left(q_0\right)$ appears cannot be used (see [11] to [15]), and one should revert to the full four-dimensional Bethe-Salpeter wavefunction (for example the Barbieri-Remiddi one [24]).

ii- In the soft photon limit, the combination $\operatorname{Im}\mathcal{I}_1\left(s,x\right)+\operatorname{Im}\mathcal{I}_2\left(s,x\right)$ behaves as a constant when $x\to 0$ despite the fact that each cut diverges in that limit (see the $1/x$ in (49) and (50)). This will guarantee that the imaginary part of the whole amplitude vanishes in the soft photon limit thanks to the presence of k_ρ in the tensor structure of (46). The following simple formula gives the behaviour of the imaginary part when $x\to 0$

$$\operatorname{Im}\mathcal{I}_1\left(s,x\right)+\operatorname{Im}\mathcal{I}_2\left(s,x\right)\overset{x\to 0}{=}\frac{\eta}{s}\frac{1}{16\pi}\left[\frac{f_B}{\sqrt{1-\frac{4m^2}{s}}}+\frac{\sqrt{s-4m^2}}{2}f_B'\right]+\mathcal{O}\left(x\right)$$

where

$$\begin{aligned}
f_B &= F_B\left(0,\mathbf{q}^2=s/4-m^2\right)\\
f_B' &= \left[\frac{\partial F_B\left(q_0,|\mathbf{q}|\right)}{\partial q_0}+\frac{1}{\sqrt{1-4m^2/s}}\frac{\partial F_B\left(q_0,|\mathbf{q}|\right)}{\partial|\mathbf{q}|}\right]_{q_0\rightarrow 0,|\mathbf{q}|\rightarrow\sqrt{s/4-m^2}}
\end{aligned}$$

To get a feeling of the underlying physics, note that for a constant form factor $F_B=F_B^{\text{Const}}$, the absorptive parts are

$$\begin{aligned}
\operatorname{Im}\mathcal{I}_1\left(s,x\right) &= \frac{\eta}{s}\frac{F_B^{\text{Const}}}{16\pi x}\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{s}}}{1-\sqrt{1-\frac{4m^2}{s}}}\right]\theta\left(s-4m^2\right)\\
\operatorname{Im}\mathcal{I}_2\left(s,x\right) &= -\frac{\eta}{s}\frac{F_B^{\text{Const}}}{16\pi x}\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{s(1-x)}}}{1-\sqrt{1-\frac{4m^2}{s(1-x)}}}\right]\theta\left(s-\frac{4m^2}{1-x}\right)
\end{aligned}$$

The relative -1 factor is responsible for the destructive interference when $x\rightarrow 0$. Explicitly, the SPL is

$$\operatorname{Im}\mathcal{I}_1\left(s,x\right)+\operatorname{Im}\mathcal{I}_2\left(s,x\right)\overset{x\rightarrow 0}{\rightarrow}\frac{\eta}{s}\frac{F_B^{\text{Const}}}{16\pi}\frac{1}{\sqrt{1-4m^2/s}}$$

iii- The dispersion integrals are

$$\begin{aligned}
\mathcal{I}_1\left(M^2,x\right) &= \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds}{s-M^2}\operatorname{Im}\mathcal{I}_1\left(s,x\right) && (52)\\
\mathcal{I}_2\left(M^2,x\right) &= \frac{1}{\pi}\int_{\frac{4m^2}{1-x}}^{\infty}\frac{ds}{s-M^2}\operatorname{Im}\mathcal{I}_2\left(s,x\right) && (53)
\end{aligned}$$

We expect that the dispersive part will inherit the SPL properties of the absorptive parts.

The oblique cuts are seen to be decreasing functions of x, and disappear when $t^2\rightarrow 0$ ($x\rightarrow 1$). Further, in that limit $\mathcal{I}_1\left(s,x\right)\rightarrow\mathcal{I}\left(s\right)$, with $\mathcal{I}\left(s\right)$ the two-photon effective form factor (32). At the decay rate level, the oblique cuts are really negligible because the phase-space spectrum $\rho\left(x,a\right)$ is highly peaked around $x=1-a$. In fact, independently of the form of $\mathcal{I}_1\left(s,x\right)$, provided it is not too badly behaved, we can set $\mathcal{I}\left(M^2,x\right)\rightarrow\mathcal{I}_1\left(M^2,1\right)$ in (47) to get an estimation for the ratio as

$$\frac{d\Gamma\left(p\text{-}Dm\rightarrow e^+e^-\gamma\right)/dx}{\Gamma\left(p\text{-}Dm\rightarrow\gamma\gamma\right)}=\frac{\alpha}{3\pi}\frac{\left|\mathcal{I}\left(M^2,x\right)\right|^2}{\left|\mathcal{I}\left(M^2,1\right)\right|^2}\rho\left(x,a\right)\approx\frac{\alpha}{3\pi}\rho\left(x,a\right)\qquad(54)$$

with $a=4m_e^2/M^2$. This is a rather universal result: the highly peaked spectrum wipes out the details of the dynamics, and the same ratio holds for other pseudoscalar ($\pi^0\rightarrow e^+e^-\gamma$, $K_S\rightarrow e^+e^-\gamma$,...). Integrated over x, this gives

$$R_{Dm}^{th}\equiv\frac{\Gamma\left(p\text{-}Dm\rightarrow e^+e^-\gamma\right)}{\Gamma\left(p\text{-}Dm\rightarrow\gamma\gamma\right)}\sim\frac{\alpha}{3\pi}\left(17.1\right)\qquad(55)$$

while for example $R_{\pi^0}^{th}\sim\frac{\alpha}{3\pi}\left(15.3\right)\sim 0.012$, in good agreement with experiments [34].

If we took some time to recall known stuff, it is to emphasize that the present discussion of $p\text{-}Dm\rightarrow e^+e^-\gamma$ is really to be taken as an illustration of some suppression effects. For the more interesting orthopositronium decay, similar kinematical effects are expected. However, comparing the respective phase-space photon spectrums (figs. 6, 10), the suppression of oblique cuts is seen to be less effective in orthopositronium.

We will now evaluate the contribution of each type of cuts for the case of the Schrödinger momentum wavefunction form factor (38a). In doing so, we will gain a better understanding of the analyticity restoration at the level of the differential decay rate for $p\text{-}Dm \to e^+e^-\gamma$.

4.2 The vertical cuts reproduce standard approach results

To precisely show what happens when the oblique cuts are forgotten, we compute here the rate with the vertical cuts only. This is the result one would obtain starting with (15).

From the first cut imaginary part (49), the decay amplitude is found through an unsubstracted dispersion relation. The calculation is very similar to that of the $p\text{-}Ps \to \gamma\gamma$ decay amplitude, and we get

$$\mathcal{I}_1\left(M^2, x\right) = \frac{C\phi_o}{M^3}\frac{1}{x}\left[\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right] \tag{56}$$

Inserting the result for $\mathcal{I}_1$ into the expression of the differential rate with $C^2 = M/m^2$ and $|\phi_o|^2 = \alpha^3 m^3/8\pi$, we get:

$$\frac{d\Gamma\left(p\text{-}Dm \to \gamma e^+e^-\right)}{dx} = \frac{\alpha^6 m}{6\pi}\left(\frac{4m^2}{M^2}\right)\left|\frac{2}{\pi}\arctan\frac{M}{2\gamma}\right|^2\frac{\rho\left(x,a\right)}{x^2}$$

Note that, independently of the value for γ, the photon spectrum is always linear in x for $x \approx 0$, due to the $1/x$ in (56). This is an incorrect soft photon behaviour since Low's theorem requires a cubic spectrum instead (the amplitude behaves as x, and an additional x comes from phase-space). The γ-dependent corrections cancel in the ratio

$$\frac{d\Gamma\left(p\text{-}Dm \to \gamma e^+e^-\right)/dx}{\Gamma\left(p\text{-}Dm \to \gamma\gamma\right)} = \frac{\alpha}{3\pi}\frac{\rho\left(x,a\right)}{x^2} \tag{57}$$

(compare with (54)). For completeness, the total rate is simply:

$$\frac{\Gamma\left(p\text{-}Dm \to \gamma e^+e^-\right)}{\Gamma\left(p\text{-}Dm \to \gamma\gamma\right)} = \frac{\alpha}{3\pi}\left[\frac{4}{3}\sqrt{1-a}\left(a-4\right) + 2\ln\left(\frac{1+\sqrt{1-a}}{1-\sqrt{1-a}}\right)\right]$$

Numerically,

$$\frac{\Gamma\left(p\text{-}Dm \to \gamma e^+e^-\right)}{\Gamma\left(p\text{-}Dm \to \gamma\gamma\right)} \approx \frac{\alpha}{3\pi}\left(18.7\right)$$

to be compared with (55).

4.3 The oblique cut contribution to the rate

We have just seen that the vertical cuts suffice to reproduce the lowest order decay rate. It is interesting to investigate how the oblique cuts can restore the analytical behaviour of the spectrum without affecting this lowest order evaluation.

Unfortunately, as mentioned earlier, the form factor to be used to compute oblique cuts imaginary parts (and thereby their contributions to the decay rate) is not the simple Schrödinger momentum wavefunction, because the energy-dependent part is not set to zero. To keep the discussion as concise as possible, we will nevertheless use this Schrödinger wavefunction, i.e.

$$F_B\left(q_0, \mathbf{q}\right) \equiv C\phi_o\frac{8\pi\gamma}{\mathbf{q}^2+\gamma^2}$$

We could say that the standard approach neglect of subleading kinetic-energy dependences is taken. In fact, since our purpose is only illustrative, the present approximation is not very important. The cancellation leading to the correct SPL can still be observed. Of course, the exact value of the oblique cut contributions to the decay rate will not be exact, but this is not a urgent question for now (the decay to $e^+e^-\gamma$ is about $1,5\%$ of the rate to $\gamma\gamma$ and the oblique cut corrections are at most of a few percent, while p- Dm has not even been observed yet!).

To be clear, basic principles (gauge invariance and analyticity) require the introduction of oblique cuts, but practically, for low order evaluations, their exact form is not relevant. On the contrary, a precise definition of the form factor is necessary to tackle seriously the calculation of o-$Ps \to \gamma\gamma\gamma$, but this is left for future work (because it may be more useful to uncover the remnants of oblique cuts in standard NRQED expansions first).

When using the Schrödinger form factor, the imaginary part of the effective form factor has the constant SPL

$$\operatorname{Im}\mathcal{I}_1(s,x) + \operatorname{Im}\mathcal{I}_2(s,x) \overset{x\to 0}{\to} \frac{\eta}{s}\frac{C\phi_o}{2}\frac{\gamma\left(s/4-\gamma^2-m^2\right)}{\sqrt{1-\frac{4m^2}{s}}\left(\gamma^2-m^2+\dfrac{s}{4}\right)^2} \tag{58}$$

We will now integrate the dispersion relation (53). Obviously, this integral is quite complicated and not very interesting for the present purpose. Instead, we revert to numerical evaluation of $\mathcal{I}_2\left(M^2,x\right)$ as a function of x, and compare it to (56).

In the figure 12, we plot the vertical cuts contribution (dashed line), normalized to get a constant as

$$\overline{\mathcal{I}_1}\left(M^2,x\right) \equiv \left(\frac{xM^3}{C\phi_o}\right)\mathcal{I}_1\left(M^2,x\right) = \frac{2}{\pi}\arctan\frac{M}{2\gamma}$$

The complete effective form factor, with the same normalization (solid line) is

$$\begin{aligned}\overline{\mathcal{I}}\left(M^2,x\right) &\equiv \left(\frac{xM^3}{C\phi_o}\right)\left[\mathcal{I}_1\left(M^2,x\right)+\mathcal{I}_2\left(M^2,x\right)\right]\\ &= \frac{2}{\pi}\arctan\frac{M}{2\gamma} + \frac{1}{\pi}\frac{xM^3}{C\phi_o}\int_{\frac{4m^2}{1-x}}^{+\infty}\frac{ds}{s-M^2}\operatorname{Im}\mathcal{I}_2(s,x)\end{aligned}$$

The plots are drawn for $m=1, M=1.9975$ (i.e. $\gamma\approx 0.05$) and for $m=1, M=1.99975$ ($\gamma\approx 0.016$). Note that the physical value $\gamma\approx m\alpha/2$ corresponds to $\gamma\approx 0.0036$. For $x\to 0$, one can verify that $\overline{\mathcal{I}}\left(M^2,x\right)\to 0$.

The figures show that near zero the two cuts interfere destructively in order to maintain a correct analytical behaviour for the whole amplitude. Away from $x=0$, the oblique cut contributions are strongly suppressed relatively to the vertical one, and this suppression increases as γ decreases. As can be seen on the graph, it is typically for $x \lesssim \gamma/m$ that the oblique cut contributes.

Therefore, we can summarize by giving a simple representation of the different contributions to the amplitude. From the figure 12, one can see that the behaviors of the vertical cuts, the oblique cuts and their combination are quite precisely modelled as

$$\mathcal{I}_1 \sim \frac{1}{x},\ \mathcal{I}_2 \sim -\frac{\gamma/M}{x\left(x+\gamma/M\right)} \Rightarrow \mathcal{I} = \mathcal{I}_1+\mathcal{I}_2 \sim \left(\frac{1}{x+\gamma/M}\right)$$

As a consequence, the spectrum behaves as

$$\frac{d\Gamma\left(p\text{-}Dm\to\gamma e^+e^-\right)}{dx} \sim \left|\mathcal{I}\right|^2\rho\left(x,a\right) \sim x^3\left(\frac{1}{x+\gamma/M}\right)^2 \tag{59}$$

i.e. a linear spectrum when $\gamma \to 0$, and a x^3 spectrum when x is small. The effect of $\gamma \neq 0$ is therefore to soften the photon spectrum and slightly reduce the total width. This behaviour is exactly what we postulated in a previous work [19].

In conclusion, the oblique cuts have a small contribution to the decay rate comparatively to the vertical cuts. However, their presence is essential to guarantee the analytical properties of the amplitude expected from Low's theorem.

4.4 Overview of $K_S \to e^+e^-\gamma$

The restoration of analyticity by a cancellation among the vertical and oblique cuts is presented in the context of the kaon decay $K_S^0 \to e^+e^-\gamma$ via a charged pion loop (see [28] to [33], and also [21]). In this analysis, the form factor describing the $K \to \pi\pi$ vertex is taken as a constant. This decay is interesting since there is a close similarity between this hadronic decay process and the present QED bound state decay. The main conclusion is that even if charged particles are present in intermediate states, the amplitude has to vanish in the soft photon limit, as expected from the fact that the decays $K_S^0 \to \gamma\gamma^*$ or p-$Dm \to \gamma\gamma^*$ involve only neutral external bosons. Further, this K_S^0 decay provides a physically sensible process where one can analyze the constant form factor assumption, since for such a loosely bound system as the p-Dm a constant form factor cannot be realistic.

Note that the modern approach to this decay channel is Chiral Perturbation Theory, see for example [29], [30].

4.4.1 The pion loop model

Details can be found in [19], so let us simply review the main steps. The decay amplitude is modeled as a charged pion loop, to which photons are attached either by the one-photon or the two-photon seagull coupling (see figure 13). Gauge invariance is thereby enforced, but is manifest only after the loop integration has been done using dimensional regularization. At this stage, the amplitude reaches a form similar to (46)

$$\begin{aligned}\mathcal{M}\left(K_S^0 \to \gamma e^+e^-\right) &= \frac{-2e^3}{(4\pi)^2}\mathcal{M}\left(K_S^0 \to \pi^+\pi^-\right)\frac{1}{m_\pi^2}\mathcal{F}\left(M^2,x\right)\times \\ &\qquad \varepsilon_\mu^*\left(k\right)\left\{\overline{u}\left(p\right)\gamma_\nu v\left(p'\right)\right\}\frac{g^{\mu\nu}\left(k\cdot t\right)-t^\mu k^\nu}{t^2}\end{aligned}$$

with $t = p + p' = P - k$, m_π the pion mass, x the reduced photon energy $2k^0/M$ and M the kaon mass. $\mathcal{F}(s,x)$ is the Feynman parameter integral

$$\mathcal{F}\left(s,x\right) = \int_0^1 dy \int_0^{1-y} dz \frac{4zy}{1-4\left(a-b\right)zy+4by\left(y-1\right)+i\varepsilon} \tag{60}$$

with the definitions $a = s/4m_\pi^2, b = s\left(1-x\right)/4m_\pi^2$ (in the K rest-frame). This Feynman parameter integral plays exactly the same role as $\mathcal{I}\left(M^2,x\right)$, i.e. that of an effective form factor (the only difference being that the momentum loop integration has been done, and the imaginary part is obtained using the prescription $i\varepsilon$, instead of by cutting propagators, this is anyway strictly equivalent). The differential rate is

$$\frac{d\Gamma\left(K_S^0 \to e^+e^-\gamma\right)/dx}{\Gamma\left(K_S^0 \to \pi^+\pi^-\right)} = \frac{\alpha^3}{3\pi^3}\frac{1}{a_\pi^2\sqrt{1-a_\pi}}\left|\mathcal{F}\left(M^2,x\right)\right|^2\rho\left(x,a_e\right)$$

with $a_\pi = 4m_\pi^2/M^2, a_e = 4m_e^2/M^2$, m_e the electron mass. The phase-space is given in (48) with the same bounds $x \in [0, 1-a_e]$.

Using the same pion loop model, the two-photon decay rate is similarly obtained, and

$$\frac{\Gamma\left(K_S^0 \to \gamma\gamma\right)}{\Gamma\left(K_S^0 \to \pi^+\pi^-\right)} = \frac{\alpha^2}{\pi^2}\frac{1}{a_\pi^2\sqrt{1-a_\pi}}\left|\mathcal{F}\left(M^2,1\right)\right|^2$$

As previously mentioned (54), the ratio of the two electromagnetic modes is approximately given by

$$\frac{d\Gamma\left(K_S^0 \to e^+e^-\gamma\right)/dx}{\Gamma\left(K_S^0 \to \gamma\gamma\right)} = \frac{\alpha}{3\pi}\frac{\left|\mathcal{F}\left(M^2,x\right)\right|^2}{\left|\mathcal{F}\left(M^2,1\right)\right|^2}\rho\left(x,a_e\right) \approx \frac{\alpha}{3\pi}\rho\left(x,a_e\right)$$

Anyway, as we will now discuss, the precise x behavior is important regarding the enforcement of the correct SPL.

4.4.2 Dispersion Relations

The effective form factor $\mathcal{F}\left(M^2,x\right)$ is calculated using dispersion relations [21], [31]. Its imaginary part consists of the vertical cuts (see figure 13)

$$\operatorname{Im}\mathcal{F}_1\left(s,x\right) = \frac{2\pi m^2}{sx^2}\left\{\frac{2m^2}{s}\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{s}}}{1-\sqrt{1-\frac{4m^2}{s}}}\right] - (1-x)\sqrt{1-\frac{4m^2}{s}}\right\}\theta\left(s-4m^2\right)$$

and the oblique cuts

$$\operatorname{Im}\mathcal{F}_2\left(s,x\right) = -\frac{2\pi m^2}{sx^2}\left\{\frac{2m^2}{s}\ln\left[\frac{1+\sqrt{1-\frac{4m^2}{s(1-x)}}}{1-\sqrt{1-\frac{4m^2}{s(1-x)}}}\right] - (1-x)\sqrt{1-\frac{4m^2}{s(1-x)}}\right\}\theta\left(s-\frac{4m^2}{1-x}\right)$$

(compare with $\operatorname{Im}\mathcal{I}\left(s,x\right)$). Using an unsubstracted dispersion relation

$$\mathcal{F}\left(M^2,x\right) = \operatorname{Re}\mathcal{F}\left(M^2,x\right) = \frac{1}{\pi}\int\frac{ds}{s-M^2}\operatorname{Im}\mathcal{F}\left(s,x\right) \tag{61}$$

for $M^2 < 4m_\pi^2$. The final expression for $\mathcal{F}$ can be obtained by analytical continuation for any value of M^2

$$\mathcal{F}\left(M^2,x\right) = -\frac{1}{2\left(a-b\right)} + \frac{1}{\left(a-b\right)^2}\left(\frac{1}{2}\left(f\left(a\right)-f\left(b\right)\right) + b\left(g\left(a\right)-g\left(b\right)\right)\right) \tag{62}$$

with

$$\begin{aligned} f\left(x\right) &= \begin{cases} \arcsin^2\left(\sqrt{x}\right) & 0<x<1 \\ -\left(\ln\left(\sqrt{x}+\sqrt{x-1}\right)-\frac{1}{2}i\pi\right)^2 & x>1 \end{cases} \\ g\left(x\right) &= \begin{cases} \sqrt{\frac{1-x}{x}}\arcsin\left(\sqrt{x}\right) & 0<x<1 \\ \sqrt{\frac{x-1}{x}}\left(\ln\left(\sqrt{x}+\sqrt{x-1}\right)-\frac{1}{2}i\pi\right) & x>1 \end{cases} \end{aligned}$$

This result agrees with that of $\mathcal{O}\left(p^4\right)$ Chiral Perturbation Theory [29], [30], or with a direct evaluation of the Feynman parameter integral (60).

4.4.3 Soft-Photon limits

Individual contributions of each cut to the imaginary part are divergent as $\sim 1/x^2$. However, their combination is finite in the soft photon limit $x \to 0$. Indeed, applying L'Hospital's rule twice, we get

$$\operatorname{Im}\mathcal{F}_1\left(s,x\right) + \operatorname{Im}\mathcal{F}_2\left(s,x\right) \overset{x\to 0}{=} \frac{2\pi m^4}{s^2}\frac{1}{\sqrt{1-4m^2/s}} \tag{63}$$

This constant SPL is not altered by the dispersion integrals, and the effective form factor behaves as a constant for very low x

$$\mathcal{F}\left(M^2, x\right) \overset{x \to 0}{\to} -\frac{a_\pi}{4} + \frac{a_\pi^2}{4\left(a_\pi - 1\right)} g\left(\frac{1}{a_\pi}\right) \tag{64}$$

(note that for $a_\pi < 1$, the imaginary part of (64) is the same as in (63)). Since there is a factor k in the tensor structure, the whole amplitude vanishes in the SPL, as expected from Low's theorem.

Let us turn to the soft photon behaviour of the differential width. Since the amplitude behave as x for low x, the resulting spectrum is in x^3 (x^2 from the squared amplitude and a x from phase space). In other words, if we had forgotten one cut, the resulting spectrum behaviour would have been divergent as $1/x$ near zero instead of vanishing like x^3. Compared to the paradimuonium and orthopositronium cases, it is even more crucial not to forget oblique cuts, because of the IR divergence that would be spuriously generated.

4.4.4 Non-constant Form factor

For a non-constant amplitude $\mathcal{M}\left(K_S^0 \to \pi^+\pi^-\right) \equiv F\left(\left(q+k\right)^2, \left(q-t\right)^2\right)$, structure-dependent terms must be supplemented to enforce gauge invariance [28]. By an analysis strictly parallel to the orthopositronium one, the origin of the problem can be traced down to the bremsstrahlung amplitudes

$$\mathcal{M}_{IB}^{\mu}\left(K_S^0 \to \pi^+\left(p_1\right)\pi^-\left(p_2\right)\gamma\left(k\right)\right) = F\left(\left(p_1+k\right)^2, p_2^2\right)\frac{p_1^\mu}{p_1 \cdot k} - F\left(p_1^2, \left(p_2+k\right)^2\right)\frac{p_2^\mu}{p_2 \cdot k}$$

The effect of the structure terms is to restore both gauge invariance and analyticity by accounting for the variations of F.

In conclusion, the pseudoscalar decay $K_S^0 \to e^+e^-\gamma$ is seen as an interesting system, being the low energy mesonic analogue of dimuonium decay to $e^+e^-\gamma$. The dispersive techniques introduced to deal with electromagnetic bound states are the standard tools used to described $K_S^0 \to e^+e^-\gamma$. Also, the same general electrodynamical principles must hold. Namely, gauge invariance and analyticity in the SPL are shown to be implemented by the same mechanism, i.e. interference between imaginary part contributions. For non-constant form factors, structure terms come into play to account for the variation of the form factor, to preserve the cancellation among contributing cuts in the SPL.

5 Conclusions

In this paper, we have used dispersion techniques to analyze positronium decay amplitudes. In this way, emphasis is put on basic principles, i.e. gauge invariance and soft photon limits. This provides interesting new insights into bound state decay dynamics. Indeed, basic principles implementation requires the non-perturbative treatment of some binding energy effects. What our work has shown is that even if NRQED scaling arguments are a very powerful calculational tool, the inherent breaking of covariance hinders the manifestation of basic principles. In other words, binding energy effects are usually perturbatively expanded along with radiative corrections in the NRQED approach. The present work can therefore be understood as complementary to NRQED, and may emphasize its underlying model-dependent assumptions.

Some consequences are worth repeating here. First, no approximations were needed to reach standard three-dimensional convolution-type decay amplitudes (except of course the neglect of oblique cuts and structure-dependent amplitudes). This provides a more concise (and consistent) scheme to deal with spin structures and energy-dependences of the Bethe-Salpeter wavefunction. As a by-product, we showed that some formulas usually quoted in the literature are approximations, missing some of those $\mathcal{O}\left(\alpha^2\right)$ corrections they are meant to evaluate.

Also, our method emphasizes the fact that the exact integration of lowest order decay amplitudes contains $\mathcal{O}\left(\alpha\right)$ corrections. This is not an artifact of our model. Rather, this is a theoretical issue; a great deal of work in positronium decay calculation being devoted to avoid double-counting. In our view, the most appropriate formulation is to consider the corrections arising at the lowest order as binding energy effects, and therefore to express them in terms of γ (see (45)). Further, this reordering of the non-relativistic perturbation series may be much more adequate to deal with QCD bound state like quarkonia, for which little is known about the wavefunction.

Concerning orthopositronium, a consistent picture of the decay process is built. Gauge invariance and analyticity are at last correctly implemented. Further, by exploiting the fact that some destructive interferences must occur among the various contributions, bounds can be extracted on their magnitudes. Unfortunately, only lower bounds are given, but one could reasonably expect that those bounds serve as estimates, and conclude that the present $\mathcal{O}\left(\alpha^2\right)$ NRQED calculation is complete. However, one should keep in mind that no proof of this conjecture exists at present.

Let us make a comment about this optimistic conclusion. We have shown that the decay process at lowest order in α requires the introduction of both subleading contributions (oblique cuts) and additional structure-dependent contributions. This requirement is to be understood at the level of basic principles. In particular, gauge invariance is valid only if structure terms are taken into account, i.e. gauge invariance does link photons inside the bound state to radiating photons (emitted in the decay). We have seen that NRQED proceeds by a perturbative reconstruction of the Bethe-Salpeter loop. If gauge invariance is taken as a constraint in this reconstruction, some contributions may be missed, because the loop is ultimately not gauge invariant. Of course, since structure terms are invoked to restore gauge invariance, and since they should be of order $\mathcal{O}\left(\gamma^3\right) \sim \mathcal{O}\left(\alpha^3\right)$, such questions are not important practically at the present level of precision. Anyway, it would be interesting to perform a thorough analysis of gauge-dependent contributions arising in NRQED, in order to get an independent estimate of structure-dependent contributions.

The central achievement of this work is to single out and characterize the structure-dependent contributions as responsible for a breakdown of standard NRQED at the $\mathcal{O}\left(\alpha^3\right)$ level, or even sooner in pathological cases. By breakdown is meant that considering only a two-body e^+e^- Bethe-Salpeter wavefunction is no longer sufficient (structure-dependent terms could in principle be obtained from a two-body to three-body Green function). One should understand positronium calculations as a two step process: the wavefunction is obtained from BS analyses, while the decay process is calculated using NRQED. So it is really the Bethe-Salpeter basis that is at stake, not the non-relativistic effective theory. In our view, theoretical advances now require going beyond the two-body Bethe-Salpeter approach. No answer to the orthopositronium lifetime puzzle could be given before the completion of such progresses.

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6 Appendix

This appendix contains the demonstration of the assertion that

Dispersion relations constructed on the vertical cut imaginary parts of the loop model reproduce standard convolution-type amplitudes.

We will particularize the discussion to the parapositronium decay into 2γ, for which there are vertical cut contributions only. Let us emphasize that the whole discussion of this section is readily extended to any para- or orthopositronium vertical cut contributions to the decay amplitude.

We first compute the imaginary part of (10) for an arbitrary initial squared mass P^2. Considering the two possible cuts (figure 2), we obtain $\operatorname{Im}\mathcal{T}$ by replacing the two propagators on each side of $\Gamma^{\mu\nu}$ by delta functions

$$\begin{aligned}\operatorname{Im}\mathcal{T}\left(P^2\right) &\equiv \operatorname{Im}\mathcal{M}^{\mu\nu}\left(p\text{-}Ps\to 2\gamma\right)\varepsilon_{1\mu}^*\varepsilon_{2\nu}^* \\ &= \int\frac{d^4q}{2\left(2\pi\right)^2}F_B\,\delta\left(\left(q-\frac{P}{2}\right)^2-m^2\right)\delta\left(\left(q+\frac{P}{2}\right)^2-m^2\right) \\ &\quad\times Tr\left\{\gamma_5\left(\not{q}-\frac{\not{P}}{2}+m\right)\Gamma^{\mu\nu}\left(\not{q}+\frac{\not{P}}{2}+m\right)\right\}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\end{aligned}$$

After a straightforward integration over q^0 and $|\mathbf{q}|$, with $P=\left(\sqrt{P^2},\mathbf{0}\right)$, we reach

$$\begin{aligned}\operatorname{Im}\mathcal{T}\left(P^2\right) &= \frac{1}{16\pi}\sqrt{1-\frac{4m^2}{P^2}}\theta\left(P^2-4m^2\right)\int\frac{d\Omega_{\mathbf{q}}}{4\pi}F_B \\ &\quad\times Tr\left\{\gamma_5\left(\not{q}-\frac{\not{P}}{2}+m\right)\Gamma^{\mu\nu}\left(\not{q}+\frac{\not{P}}{2}+m\right)\right\}\varepsilon_{1\mu}^*\varepsilon_{2\nu}^*\end{aligned}$$

In the course of the derivation, the delta functions forced $q^0=0$ and $|\mathbf{q}|=\sqrt{P^2/4-m^2}$. In other words, the electron momenta are

$$\frac{1}{2}P\pm q=\left(\sqrt{\frac{P^2}{4}},\pm\mathbf{q}\right)\text{ with }\left(\frac{1}{2}P\pm q\right)^2=\frac{P^2}{4}-|\mathbf{q}|^2=m^2 \tag{65}$$

This kinematics is to be understood in the trace evaluation. The angular dependence arises from the relative orientations of $\mathbf{q}$ and the photon momentum $\mathbf{l}_1$. Note also that the relation (65) cannot be satisfied for the physical value $P^2=M^2<4m^2$. This is obvious since the loop cannot have an imaginary part for the physical bound states, its constituents being always off-shell. From the kinematics (65) one can prove that the factors on both sides of $\Gamma^{\mu\nu}$ are true projectors, which serve to enforce gauge invariance in the expression

$$\left(\not{q}-\frac{\not{P}}{2}+m\right)\Gamma^{\mu\nu}\left(\not{q}+\frac{1}{2}\not{P}+m\right)$$

Indeed, those two projectors play exactly the same role as external spinors when demonstrating Ward identities.

The real part will now be calculated using an unsubstracted dispersion relation

$$\operatorname{Re}\mathcal{T}\left(M^2\right)=\frac{P}{\pi}\int_{4m^2}^{+\infty}\frac{ds}{s-M^2}\operatorname{Im}\mathcal{T}\left(s=P^2\right) \tag{66}$$

where it is understood that P^2 should be replaced by s everywhere, i.e. scalar products that will appear when evaluating the trace should be expressed with the kinematics defined for an initial energy s. Since $M^2 < 4m^2$, the principal part can be omitted and $\mathcal{T}\left(M^2\right) = \mathrm{Re}\,\mathcal{T}\left(M^2\right)$. Now let us write the form factor in the general form

$$F_B \equiv C\phi_o \mathcal{F}\left(\mathbf{q}^2\right)\left(\mathbf{q}^2+\gamma^2\right) = C\phi_o\mathcal{F}\left(s/4-m^2\right)\cdot\left(s-M^2\right)/4 \qquad (67)$$

with $\gamma^2 \equiv m^2 - M^2/4$ and ϕ_o the bound state wavefunction at zero separation. Then (66) can be written as

$$\begin{aligned}\mathcal{T}\left(M^2\right) &= C\phi_o\int_{4m^2}^{+\infty} ds \int \frac{d\Omega_{\mathbf{q}}}{4\pi}\mathcal{F}\left(s/4-m^2\right)\frac{\sqrt{1-4m^2/s}}{64\pi^2} \\ &\quad \times Tr\left\{\gamma_5\left(\not{q}-\frac{\not{P}}{2}+m\right)\Gamma^{\mu\nu}\left(\not{q}+\frac{\not{P}}{2}+m\right)\right\}\varepsilon^*_{1\mu}\varepsilon^*_{2\nu}\end{aligned}$$

Let us transform the s integral back into a $|\mathbf{q}|$ integral, keeping in mind the constraints obtained when extracting the imaginary part. Using $\mathbf{q}^2 = s/4 - m^2$, $ds = 8\,|\mathbf{q}|\,d\,|\mathbf{q}|$, the decay amplitude dispersion integral is

$$\mathcal{T}\left(M^2\right) = \frac{C}{2}\phi_o\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{\mathcal{F}\left(\mathbf{q}^2\right)}{\sqrt{P^2\left(\mathbf{q}\right)}}Tr\left\{\gamma_5\left(\not{q}-\frac{\not{P}\left(\mathbf{q}\right)}{2}+m\right)\Gamma^{\mu\nu}\left(\not{q}+\frac{\not{P}\left(\mathbf{q}\right)}{2}+m\right)\right\}\varepsilon^*_{1\mu}\varepsilon^*_{2\nu}$$

where, as the notation suggests, it is understood that any P^2 appearing in the amplitude must be replaced by $4\,|\mathbf{q}|^2 + 4m^2$. In particular, $\sqrt{P^2\left(\mathbf{q}\right)}$ can be replaced by $2E_{\mathbf{q}}$ with $E_{\mathbf{q}} = \sqrt{|\mathbf{q}|^2+m^2}$. This amounts to consider the scattering amplitude with incoming on-shell electron-positron having momenta $\left(\frac{1}{2}P\left(\mathbf{q}\right)\pm q\right)^2 = m^2$ (since $q^0 = 0$). Note the fact that $E_{\mathbf{q}} > M/2$, apparently the energy is not conserved. This is not surprising since the present formula is a dispersion integral, done along the cut where $P^2\left(\mathbf{q}\right) > 4m^2$. Finally, in view of the kinematics, we introduce $k = \frac{1}{2}P\left(\mathbf{q}\right)+q$ and $k' = \frac{1}{2}P\left(\mathbf{q}\right)-q$ (hence $E_k = E_{k'} = E_{\mathbf{q}}$ and $\mathbf{k} = -\mathbf{k}' = \mathbf{q}$) to write the amplitude simply as

$$\mathcal{T}\left(M^2\right) = \frac{C}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3\,2E_{\mathbf{k}}}\left[\phi_o\mathcal{F}\left(\mathbf{k}^2\right)\right]Tr\left\{\gamma_5\left(-\not{k}'+m\right)\Gamma^{\mu\nu}\left(k,k',l_1\right)\left(\not{k}+m\right)\right\}\varepsilon^*_{1\mu}\varepsilon^*_{2\nu} \qquad (68)$$

where $\Gamma^{\mu\nu}\left(k,k',l_1\right)$ is the amplitude for on-shell $e^-\left(k\right)e^+\left(k'\right)$ scattering into 2γ. Gauge invariance is present due to the two projectors, well defined since $k^2 = k'^2 = m^2$. This ends our demonstration, and $\mathcal{T}\left(M^2\right) = \mathcal{M}\left(p\text{-}Ps \to \gamma\gamma\right)$.

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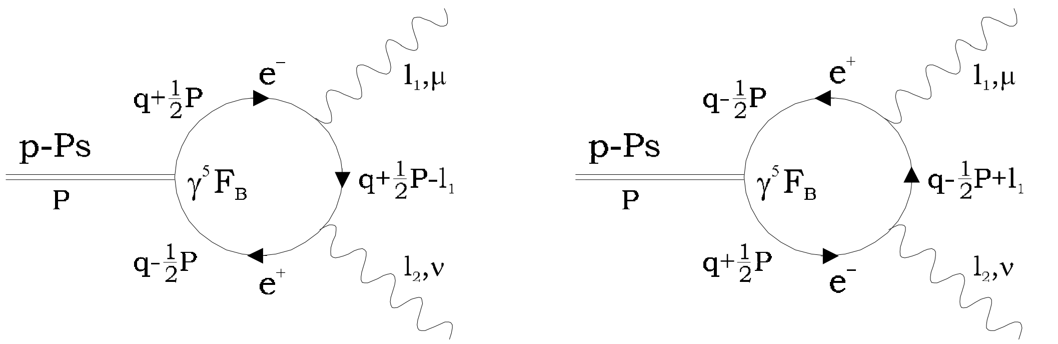

Figure 1 : The loop model direct and crossed diagram for the decay p-$Ps \to \gamma\gamma$

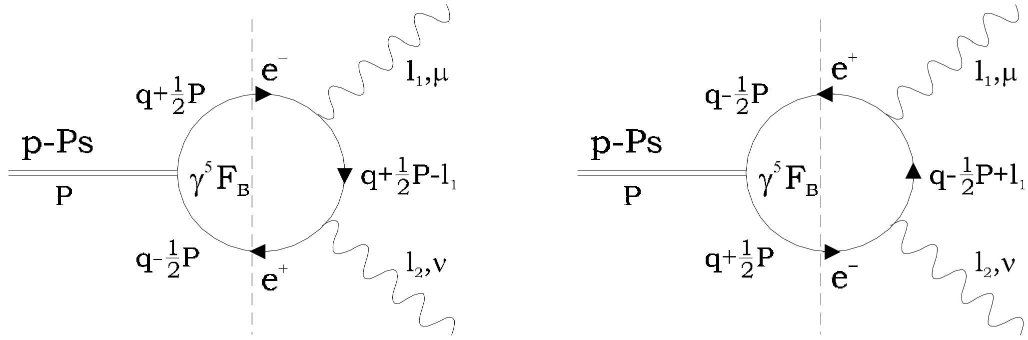

Figure 2 : The two cuts contributing to the imaginary part

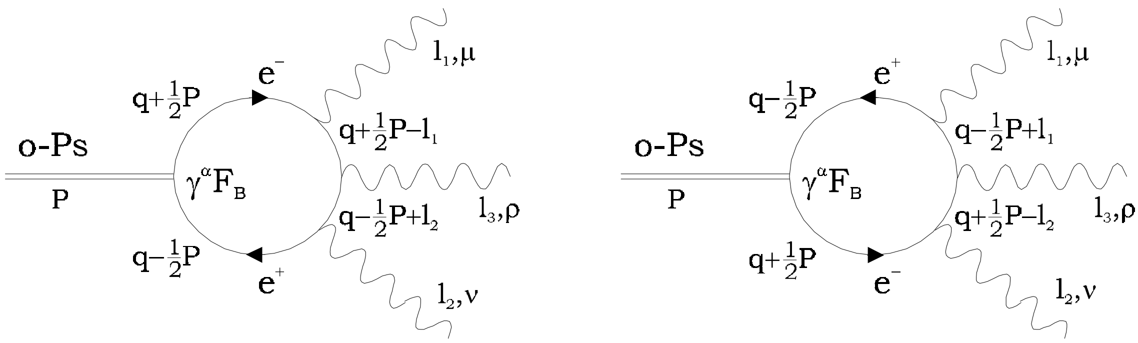

Figure 3 : The loop model diagrams $\Gamma_{12}^{\mu\nu\rho}$ for the decay o-$Ps \to \gamma\gamma\gamma$

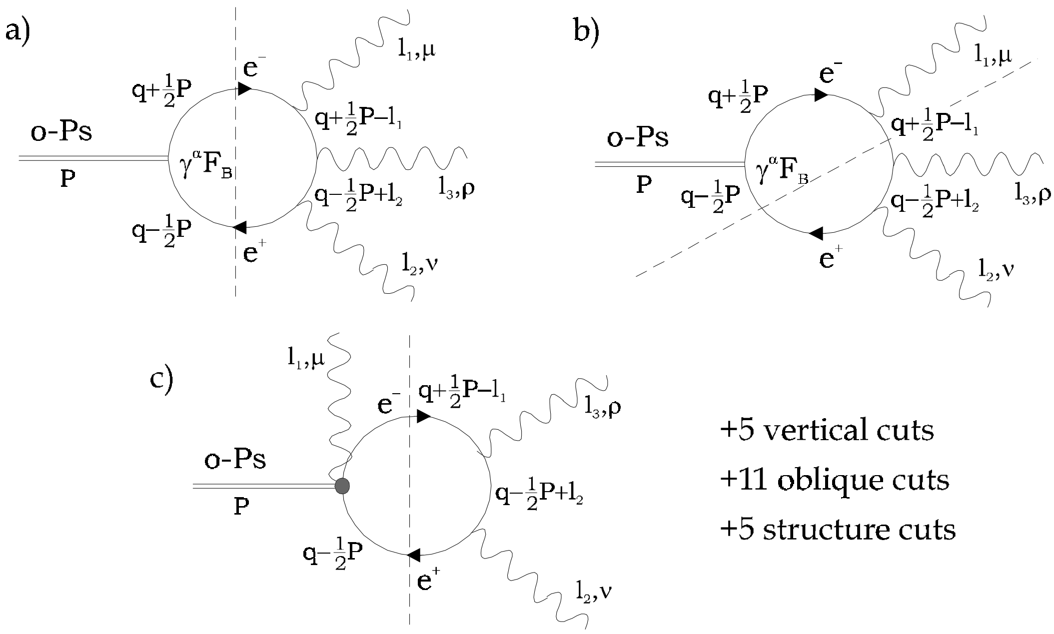

Figure 4 : Imaginary parts Im $\mathcal{V}$ (a), Im $\mathcal{D}$ (b) and Im $\mathcal{S}$ (c)

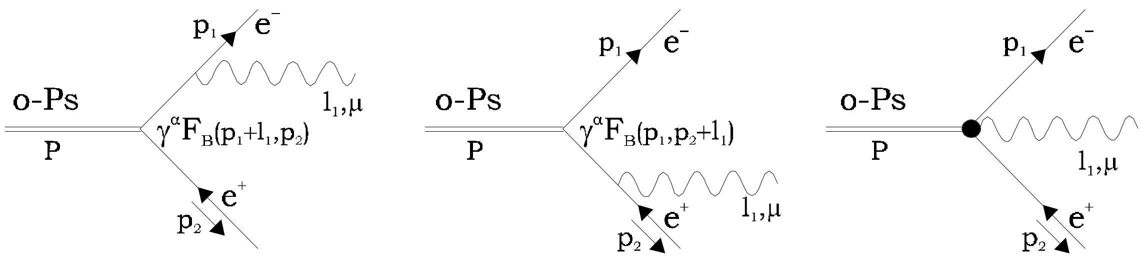

Figure 5 : Bremsstrahlung processes and structure-dependent contribution

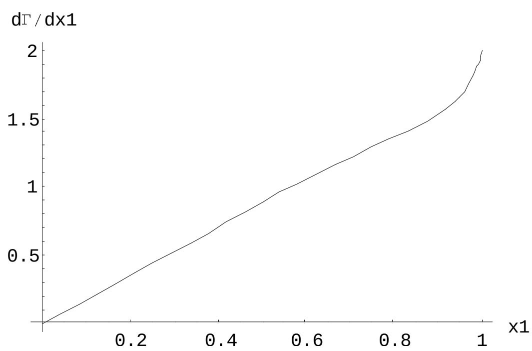

Figure 6 : Photon energy spectrum for the decay $o\text{-}Ps \to \gamma\gamma\gamma$.

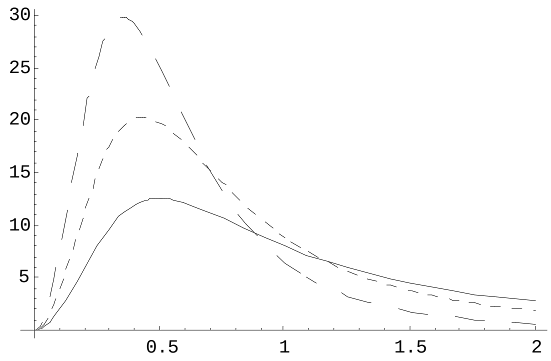

Figure 7 : Form factors $\mathbf{q}^2\mathcal{F}(\mathbf{q}^2)$, as functions of $\mathbf{q}^2$ with $\gamma = 0.5$ ($\mathcal{F}_I$ plain, $\mathcal{F}_{II}$ dashed and $\mathcal{F}_{III}$ small dashed lines)

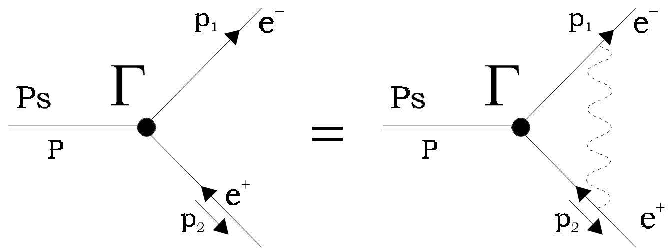

Figure 8 (a): BS vertex equation (Coulomb photon ladder approximation)

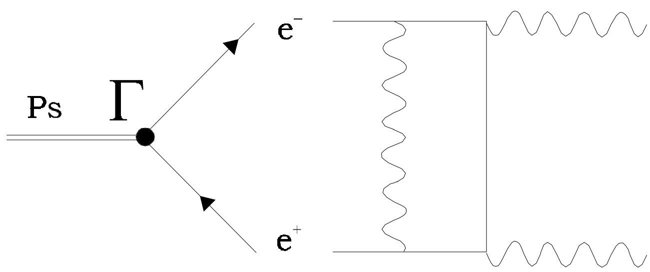

Figure 8 (b): The binding graph.

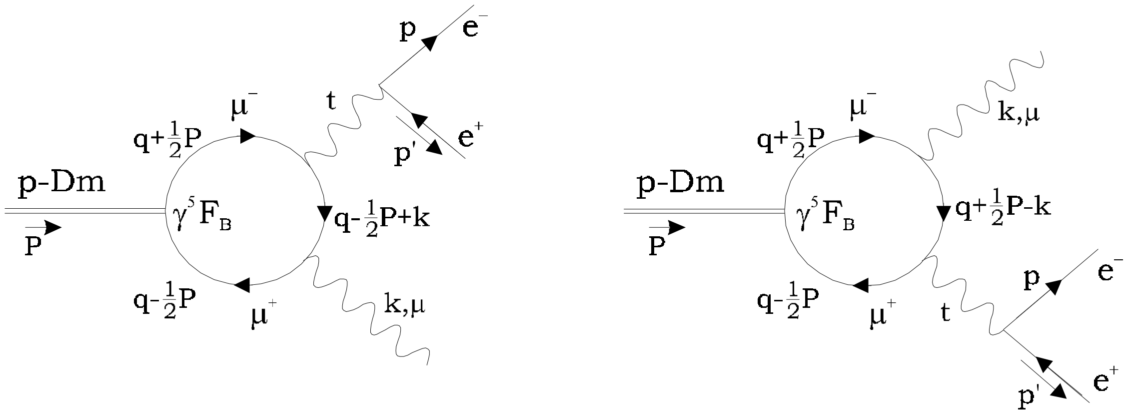

Figure 9 : The loop model amplitudes for $p\text{-}Dm \to e^+e^-\gamma$

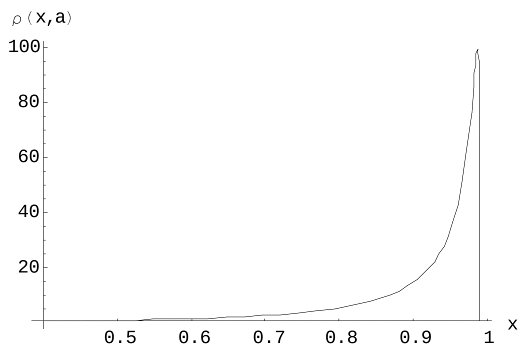

Figure 10 : The photon energy spectrum $\rho(x, a)$ for $a = 0.01$

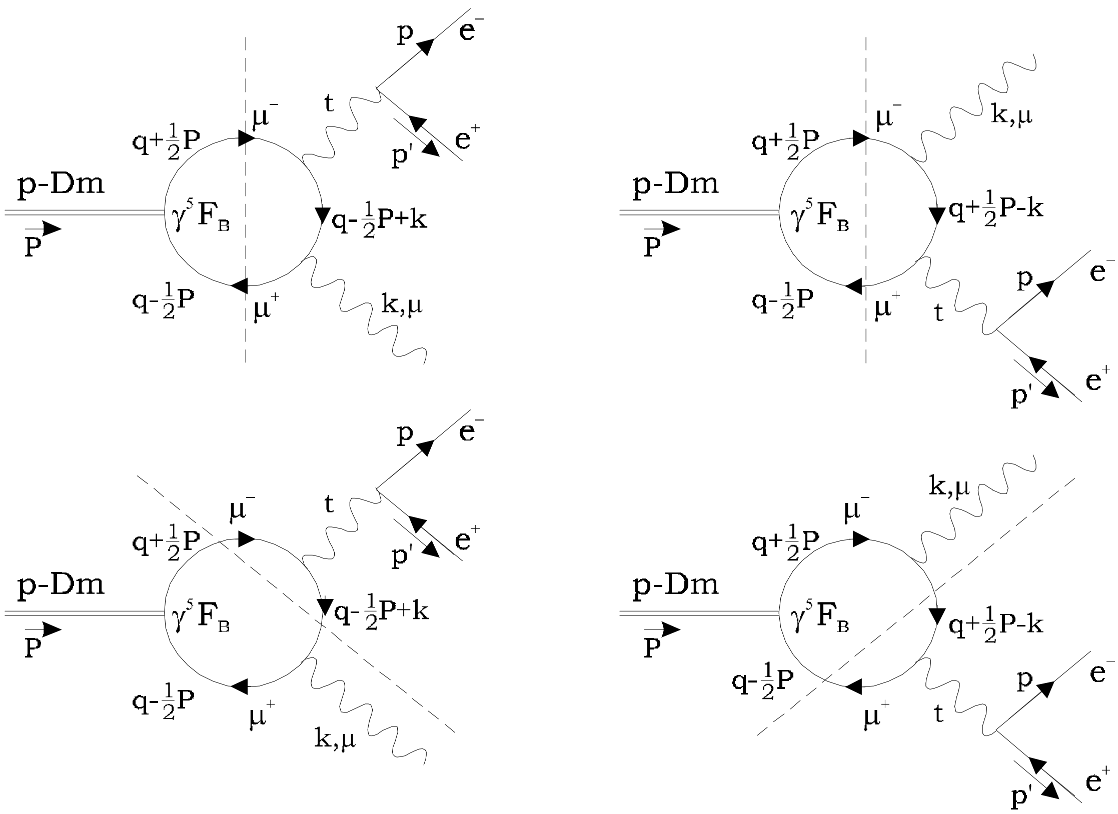

Figure 11 : Vertical and oblique cuts imaginary parts

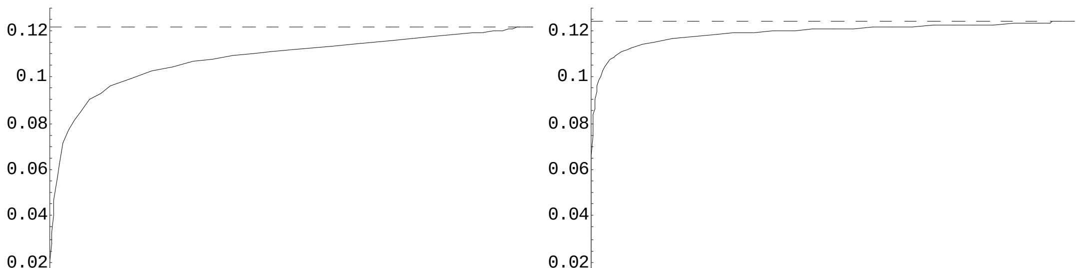

Figure 12 : The vertical cuts (dashed line) and vertical plus oblique cuts (plain line) contributions, for $\gamma \approx 0.05$ and $\gamma \approx 0.016$, as functions of the photon energy x.

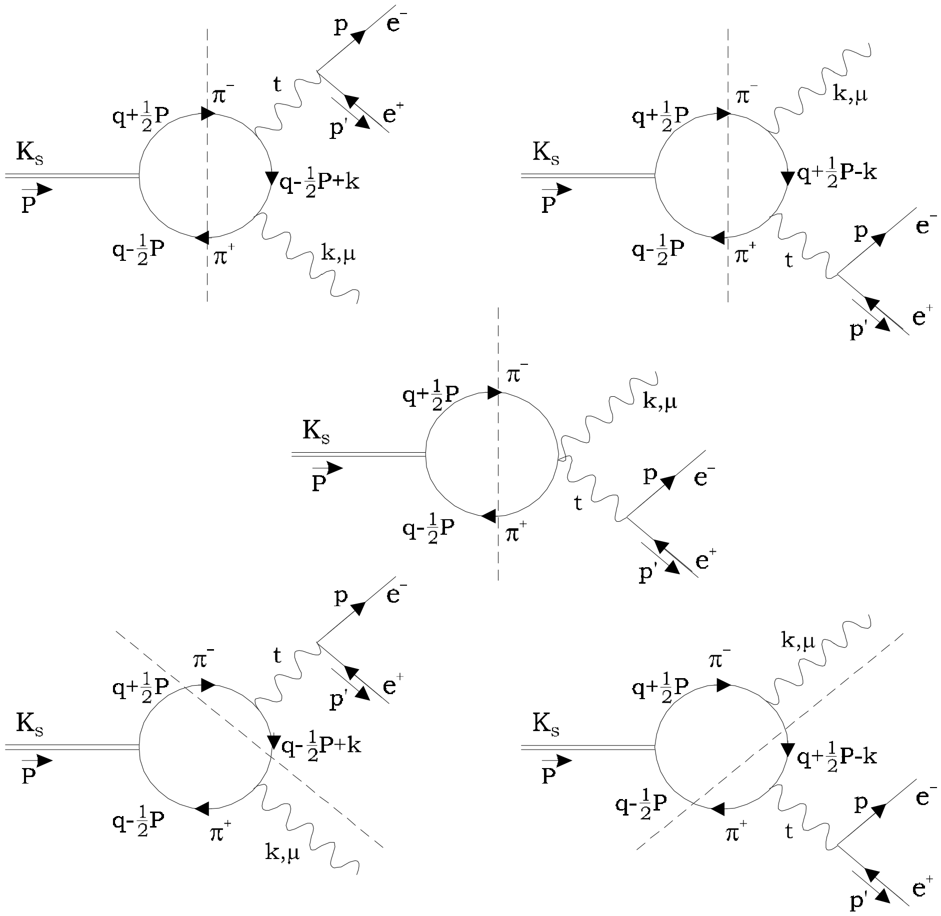

Figure 13 : Vertical and oblique cuts for the imaginary part of $K_S \to e^+e^-\gamma$